# EXTENSIONS OF $L^p$-MULTIPLIERS

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# CERTIFICATE

It is certified that the work contained in the thesis entitled, "**EXTENSIONS OF $L^p$-MULTIPLIERS**", by Parasar Mohanty, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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# Contents

# Synopsis

In this thesis we investigate some aspects of the relationship between the $L^p$-multipliers on the groups $\mathbb{R}$, $\mathbb{T}$, and $\mathbb{Z}$. The starting point of the work done in this thesis lies in the works of Asmar, Berkson, and Gillespie, G. Weiss and his co-authors. We begin with some definitions:

**Definition 0.1** *Let $G$ be a locally compact abelian group and let $\hat{G}$ be its dual. Let $\phi \in L^\infty(\hat{G})$. We say that $\phi$ is an $L^p(G)$-multiplier if the operator $T_\phi : L^p(G) \to L^p(G)$, defined by $(T_\phi f)^\wedge = \phi \hat{f}$ $\forall f in L^p(G) \cap L^2(G)$ is a bounded linear operator, and $M_p(\hat{G})$ denotes the set of all $L^p(G)$-multipliers.*

In this thesis we are only concerned with the groups $G = \mathbb{R}$ or $\mathbb{T}$ (then $\hat{\mathbb{R}}$ is identified with $\mathbb{R}$ and $\hat{\mathbb{T}}$ with $\mathbb{Z}$). We identify the circle group $\mathbb{T}$ with $[0, 1)$ with addition modulo 1. For $f \in L^1(\mathbb{R})$ our definition o f Fourier transform is

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

In this thesis we will deal with the 'extensions' of multipliers from $\mathbb{T}$ to $\mathbb{R}$ obtained by a summability kernel method which is described below.

**Definition 0.2** *A bounded measurable function $\Lambda$ on $\mathbb{R}$ is said to be a summability kernel if for each $\phi \in M_p(\mathbb{Z})$ the function*

$$W_{\phi,\Lambda}(\xi) = \sum_{n \in \mathbb{Z}} \phi(n) \Lambda(\xi - n) \tag{1}$$

*is defined pointwise a.e. and belongs to* $M_p(\hat{\mathbb{R}})$ *with* $||W_{\phi,\Lambda}||_{M_p(\hat{\mathbb{R}})} \leq C_{p,\Lambda}||\phi||_{M_p(\mathbb{Z})}$, *where the constant* $C_{p,\Lambda}$ *depends only upon* $p$ *and* $\Lambda$.

*Let* $S_p(\mathbb{R})$ *denote the set of all summability kernels.*

By an 'extension' problem we mean the construction of an $L^p(\mathbb{R})$ multiplier $W_{\phi,\Lambda}$ from an $L^p(\mathbb{T})$ multiplier $\phi$ as above. Let us also define the following set.

$S_p^0(\mathbb{R}) = \{\Lambda \in L^\infty(\hat{\mathbb{R}})$ : For each finitely supported $\phi \in M_p(\mathbb{Z})$, the function $W_{\phi,\Lambda}$ (as in Eqn.1) belongs to $M_p(\hat{\mathbb{R}})$ and there exists a constant $C_{p,\Lambda}$ (which depends only on $p$ and $\Lambda$) such that $||W_{\phi,\Lambda}||_{M_p(\hat{\mathbb{R}})} \leq C_{p,\Lambda}||\phi||_{M_p(\mathbb{Z})}\}$.

In the second chapter we first characterize $S_p^0(\mathbb{R})$ for $p = 1$ and for $p = 2$. Our result is

**Proposition 0.1** *(i)* $S_2^0 = S_2 = \{\Lambda \in L^\infty(\hat{\mathbb{R}}) : \operatorname{ess\,sup} \sum_{n\in\mathbb{Z}} |\Lambda(\xi + n)| = \delta_\Lambda < \infty\}$.

*(ii)* $S_1^0 = \{\Lambda \in L^1(\mathbb{R})^\wedge : \Lambda = \hat{F}$ with $\operatorname{ess\,sup} \sum_{n\in\mathbb{Z}} |F(x + n)| = \delta_F < \infty\}$.

While the two conditions appearing in Proposition 0.1 appear very different, a deeper analysis suggests a unified picture, which then gives us a necessary condition for $\Lambda$ to be in $S_p(\mathbb{R})$ for other values of $p$. Define

$\mathcal{F}_p = \{\Lambda \in L^\infty(\hat{\mathbb{R}})$ : for a.e. $x \in [0, 1)$, $\Lambda_x^\# \in M_p(\mathbb{T})$ with $||\Lambda_x^\#||_{M_p(\mathbb{T})} \in L^\infty[0, 1)\}$,

where $\Lambda_x^\#$ is the 1-periodic extension of $\Lambda_x = e^{2\pi i x \cdot}\Lambda$.

We prove:

**Proposition 0.2** $S_p(\mathbb{R}) \subseteq \mathcal{F}_p$ for $1 \leq p \leq 2$.

We do not have a complete characterization of the set $S_p(\mathbb{R})$ of all summability kernels. However the following result is proved:

**Theorem 0.1** *(i) If* $\Lambda_1$, $\Lambda_2 \in \mathcal{F}_p \cap (L^1(\mathbb{R}))^\wedge$ *then* $\Lambda = \Lambda_1\Lambda_2 \in S_p^0(\mathbb{R})$.

*(ii) If in addition either* $\delta_{\Lambda_1} < \infty$ *or* $\delta_{\Lambda_2} < \infty$ *then* $\Lambda \in S_p(\mathbb{R})$.

In all the previously known classes ( [29], [11], [1]) of summability kernels $\Lambda$, either $\Lambda$ has compact support or $\Lambda = \hat{F}$ for some $F \in L^1(\mathbb{R})$ having compact support. In this chapter we have a large class of summability kerne ls which do not satisfy either of the above two conditions.

In the last section of this chapter, we restrict ourselves to the case $p = 1$. From the characterization $M_1(\hat{G}) \simeq M(G)$, for $\phi \in M_1(\mathbb{Z})$ we have $\phi = \hat{\nu}$ for some $\nu \in M(\mathbb{T})$. Further, if $\Lambda \in S_1(\mathbb{R})$ then $W_{\phi,\Lambda} = \hat{\mu}$ for some $\mu \in M(\mathbb{R})$. Here we study the properties of $\nu$ which are carried over to $\mu$ by this process of extension. We denote $\mathcal{F}_0 = \{\Lambda \in S_1^0(\mathbb{R}) : \delta_\Lambda < \infty\}$. Then we first prove the following theorem:

**Theorem 0.2** *Let* $\Lambda \in \mathcal{F}_0$, $\nu \in M(\mathbb{T})$, *define* $\hat{\mu}(\xi) = W_{\hat{\nu},\Lambda}(\xi) = \sum_n \hat{\nu}(n)\Lambda(\xi - n)$, *(here* $\mu \in M(\mathbb{R})$*).*

*(a) If* $\nu$ *is an absolutely continuous measure on* $\mathbb{T}$, *then* $\mu$ *is an absolutely continuous measure on* $\mathbb{R}$ *(both with respect to the Lebesgue measure).*

*(b) If* $\nu$ *is a discrete measure, then either* $\mu \equiv 0$ *or* $\mu$ *is a discrete measure.*

For continuous measures we need an additional condition on $\Lambda$. We proved the following result by using Wiener's lemma.

**Theorem 0.3** *Let* $\Lambda \in S_1(\hat{\mathbb{R}})$, *and suppose that supp* $\Lambda$ *is compact. Then if* $\nu$ *is a continuous measure, so is* $\mu$.

We can relax the condition that *supp* $\Lambda$ is compact by another condition.

**Theorem 0.4** *Suppose* $\Lambda \in \mathcal{F}_0$ *and that* $\Lambda$ *has a decreasing radial* $L^1$*-majorant* $\Lambda_1$. *Then* $\mu$ *is a continuous measure if* $\nu$ *is.*

In the next chapter we have studied a different kind of extension. Here we prove that in fact **every** $l_p(\mathbb{Z})$ sequence can be extended to give an element of $M_q(\hat{\mathbb{R}})$ for certain values of $q$ depending on $p$.

**Theorem 0.5** *Let $S \in L^1(\mathbb{R})$, supp $S \subseteq [\frac{1}{4}, \frac{3}{4}]$ and $\sum_n |(S^\#)^\wedge(n)|^p < \infty$ for $1 < p < \infty$. Define $W_{\phi,\dot{S}}(\xi) = \sum_{n \in \mathbb{Z}} \phi(n)\dot{S}(\xi - n)$ for $\phi \in l_{p'}(\mathbb{Z})$. Then*

$$W_{\phi,\dot{S}} \in M_q(\dot{\mathbb{R}}) \ \ for \begin{cases} q \in [\frac{2p}{3p-2}, \frac{2p}{2-p}] \ \ if\ 1 < p < 2 \\ q \in [\frac{2p}{p+2}, \frac{2p}{p-2}] \ \ if\ 2 < p < \infty. \end{cases}$$

*For $p = 2$, $W_{\phi,\dot{S}} \in M_q(\dot{\mathbb{R}})$ for $1 \leq q < \infty$. Moreover,*

$$\|W_{\phi,\dot{S}}\|_{M_q(\dot{\mathbb{R}})} \leq C\tau_p\|\phi\|_p$$

*where $\tau_p = (\sum_n |(S^\#)^\wedge(n)|^p)^{\frac{1}{p}}$ and $C$ is a constant which depends only on $p$.*

This result uses multilinear interpolation. Further we also prove:

**Proposition 0.3** *Let $1 < p < 2$. Suppose $S \in L^p(\mathbb{R})$ and has compact support. If $\phi \in l_p(\mathbb{Z})$ then $W_{\phi,\dot{S}} \in M_q(\dot{\mathbb{R}})$ for $1 \leq q < \infty$ and $\|W_{\phi,\hat{S}}\|_{M_q(\hat{\mathbb{R}})} \leq C\|\phi\|_p\|S\|_p$.*

Finally we prove a maximal inequality for Theorem 0.5.

In the last chapter, we study "extensions" by summability kernels for weak-type $(p,p)$ multipliers, defined below.

**Definition 0.3** *Let $G$ be a locally compact abelian group with Haar measure $\lambda$. We say that $\phi \in L^\infty(\hat{G})$ is a weak type $(p,p)$ multiplier, if the operator $T_\phi$ on $L^p(G)$ defined by $(T_\phi f)^\wedge = \phi\hat{f} \ \ \forall f \in L^p(G) \cap L^2(G)$ satisfies*

$$\lambda\{x : |T_\phi f(x)| > t\} \leq \left(\frac{C}{t}\|f\|_p\right)^p \qquad \forall\, t > 0. \tag{2}$$

Let $M_p^{(w)}(\hat{G})$, for $1 \leq p < \infty$ denote the space of all multipliers of weak type $(p,p)$, and $N_p^{(w)}(\phi)$ is the smallest constant $C \geq 0$ such that Eqn.(2) holds.

The "restriction" problem for weak type multipliers has been studied extensively by Asmar, Berkson, and Gillespie, Bourgain, Raposo... in a series of papers. We first make a comprehensive survey of these results in the first section of this chapter.

In the next section we have proved the following weak-type analogue of the corresponding result of Berkson, Paluszyński and Weiss [11] for strong type multipliers. Namely

**Theorem 0.6** [11] *For $1 \leq p < \infty$ let $\Lambda \in M_p(\hat{\mathbb{R}})$ and let support of $\Lambda$ be compact. If $\phi \in M_p(\mathbb{Z})$ then $W_{\phi,\Lambda} \in M_p(\hat{\mathbb{R}})$. Moreover $\|W_{\phi,\Lambda}\|_{M_p(\hat{\mathbb{R}})} \leq C\|\phi\|_{M_p(\mathbb{Z})}\|\Lambda\|_{M_p(\hat{\mathbb{R}})}$, where the constant $C$ depends on $p$ and supp $\Lambda$.*

For the proof of the above theorem in [11] Berkson, Paluszyński and Weiss extended the original transference methods and introduced transference couples. We prove weak-type inequalities in a suitable context of transference couples a nd using this we prove:

**Theorem 0.7** *Let $1 < p < \infty$. Suppose $\Lambda \in M_p(\hat{\mathbb{R}^N})$ and supp $\Lambda \subseteq [\frac{1}{4}, \frac{3}{4}]^N$; for $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ define $W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}^N} \phi(n)\Lambda(\xi - n)$ on $\mathbb{R}^N$, then $W_{\phi,\Lambda} \in M_p^{(w)}(\hat{\mathbb{R}}^N)$ and $N_p^{(w)}(W_{\phi,\Lambda}) \leq C_p N_p^{(w)}(\phi)\|\Lambda\|_{M_p(\mathbb{R}^N)}$. where $C_p$ is a constant depending on $p$.*

Further, the condition that *supp* $\Lambda \subseteq [\frac{1}{4}, \frac{3}{4}]^N$ is not really crucial for the above result. If *supp* $\Lambda \subseteq [-M, M]^N$, a partition of identity argument allows a more general theorem as in [11]. Alternatively we can also do this by means o f the following lemma, which seems interesting on its own.

**Lemma 0.1** *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be a nonsingular linear transformation such that $A(\mathbb{Z}^N) \subseteq \mathbb{Z}^N$. Denote $A^t = B$. For $\phi \in l_\infty(\mathbb{Z}^N)$ define*

$$\psi(n) = \phi(Bn)$$

*and*

$$\eta(n) = \begin{cases} \phi(B^{-1}n) & n \in B\mathbb{Z}^N \\ 0 & \text{otherwise.} \end{cases}$$

*Then*

*(i) If $\phi \in M_p(\mathbb{Z}^N)$ then $\psi, \eta \in M_p(\mathbb{Z}^N)$ with multiplier norms not exceeding the multiplier norm of $\phi$.*

*(ii) If $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ then $\psi, \eta \in M_p^{(w)}(\mathbb{Z}^N)$ with weak multiplier norms not exceeding the weak multiplier norm of $\phi$.*

A special case of this lemma, corresponding to multiplication by 2 is in [29], and in [5] for weak type multipliers. For general dilation matrices $A$ as in lemma 0.1, the ideas of the proof occur in wavelet theory.

Finally as an application of Theorem 0.7 we have the following weak-type analogue of deLeeuw's [39] result.

**Theorem 0.8** *For $1 < p < \infty$ , and $\epsilon > 0$; let $\{\phi_\epsilon\} \subseteq M_p^{(w)}(\mathbb{Z})$ satisfy*

*(i)* $\lim_{\epsilon \to 0} \phi_\epsilon([\frac{x}{\epsilon}]) = \phi(x)$ *a.e.*

*(ii)* $\sup_\epsilon N_p^{(w)}(\phi_\epsilon) = K < \infty$.

*Then $\phi \in M_p^{(w)}(\mathbb{R})$ and $N_p^{(w)}(\phi) \leq \sup_\epsilon N_p^{(w)}(\phi_\epsilon)$.*

# Chapter 1

# Introduction

In the study of Fourier series, it becomes important to describe those sequences $\{c_n\}$ for which $\sum_{n \in \mathbb{Z}} c_n a_n e^{2\pi i n t}$ is always a Fourier series of a one-periodic integrable function whenever $\sum_n a_n e^{2\pi i n t}$ is such a Fourier series. Quite naturally, this problem does not remain confined to functions on the circle group $\mathbb{T}$. For locally compact abelian groups $G$ the analogous problem is to describe bounded measurable functions $\phi$ for which $\phi \hat{f}$ is always the Fourier transform of a function $g \in L^p(G)$ whenever $f \in L^p(G) \cap L^2(G)$ (see the notation below). It turns out that any such function gives rise to a bounded linear operator on $L^p(G)$ which commutes with the translations of $G$ and conversely, every translation invariant operator on $L^p(G)$ is associated with a bounded measurable function $\phi$ on $\hat{G}$ as above.

The multiplier problem is to characterize all such operators for $1 \leq p < \infty$. Except for the cases $p = 1$ and $p = 2$, this is a difficult problem even though some abstract characterizations exist. We refer to [31], [26] for details.

Before proceeding further with this introduction , we fix some notation and along the way we will state some well known results.

## 1.1 Notation and Preliminaries

$G$ will denote a locally compact abelian group and $\hat{G}$ its dual group. $G$ is equipped with a Haar measure and we will denote the integration with respect to this measure by $dx$. For $1 \leq p < \infty$, $L^p(G)$ is the Banach space of equivalence classes of complex valued measurable functions on $G$ whose $p^{th}$ power is integrable with respect to the Haar measure. The Banach space of equivalence classes of essentially bounded complex valued measurable functions on $G$ will be denoted by $L^\infty(G)$. The norms on these spaces are given by

$$\|f\|_p = (\int_G |f(x)|^p dx)^{\frac{1}{p}} \quad (1 \leq p < \infty)$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in G} |f(x)| \quad (p = \infty).$$

$C(G)$ denotes the space of continuous complex valued functions on $G$. $C_0(G)$ and $C_c(G)$ are the subspaces of $C(G)$ of functions which vanish at infinity and have compact support respectively.

$C_0(G)$ is a Banach space with the norm

$$\|f\|_\infty = \sup_{x \in G} |f(x)|.$$

If $f \in L^1(G)$ and $g \in L^p(G)$ the convolution is defined as

$$f * g(x) = \int_G f(xy^{-1})g(y)dy.$$

The space $L^1(G)$ is a commutative Banach algebra with the convolution as product.

It is well known that $L^1 * L^p(G) = L^p(G)$ for $1 \leq p < \infty$ and $L^p * L^{p'} \subset C_0(G)$, where $p'$ is the conjugate index of $p$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

The Fourier transform $\hat{f}$ of $f$ in $L^1(G)$ is given by

$$\hat{f}(\gamma) = \int_G f(x)\gamma(x^{-1})dx \quad \gamma \in \hat{G}.$$

In the case of $G = \mathbb{R}^N$ or $\mathbb{T}^N$ we use the notation

$$\hat{f}(\xi) = \int_G f(x)e^{-2\pi i x.\xi}dx \quad \forall \xi \in \hat{G}.$$

We identify $\mathbb{T}$ with the interval $[0,1)$ with addition modulo 1. If $G = \mathbb{R}^N$, $\hat{G}$ is identified with $\mathbb{R}^N$ and for $G = \mathbb{T}^N, \hat{G} = \mathbb{Z}^N$. For $G = \mathbb{Z}^N$ we denote $L^p(G)$ by $l_p(\mathbb{Z}^N)$.

The space $\mathcal{S}(\mathbb{R}^N)$ of Schwartz class functions is defined to be the class of all those $C^\infty$ functions $f$ on $\mathbb{R}^N$ (i.e. all the partial derivatives of $f$ exist and are continuous) such that

$$\sup_{x \in \mathbb{R}^N} |x^\alpha (D^\beta f)(x)| < \infty$$

for all n- tuples $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $\beta = (\beta_1, \ldots, \beta_N)$ of nonnegative integers, where for an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_N)$ we let $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$, $D^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_N}}{\partial x_1^{\alpha_1} \ldots x_N^{\alpha_N}}$.

The space of all bounded complex valued regular Borel measures on $G$ is denoted by $M(G)$. $M(G)$ is a Banach space with the norm $\|\mu\| = |\mu|(G)$ where $|\mu|$ is the total variation of $\mu$. $M(G)$ is also a commutative Banach algebra under convolution defined by

$$\mu * \nu(E) = \int_G \mu(E - x) d\nu(x).$$

For a function $f$ on $G$, and $x \in G$, the translation operator $\tau_x$ is defined by

$$\tau_x f(y) = f(x^{-1}y) \text{ for every } y \in G.$$

We end this section with the statement of the Multilinear Riesz-Thorin Interpolation theorem.

**Theorem 1.1** [42] *Let $S$ and $S_i$, $i = 1, 2 \ldots n$, be measure spaces with measures $\nu$ and $\nu_i$, $i = 1, 2, \ldots, n$, respectively. Suppose $T$ is a multilinear mapping from the product space $\mathcal{S}(\nu_1) \times \mathcal{S}(\nu_2) \times \cdots \times \mathcal{S}(\nu_n)$ to the vector space of $\nu$-measurable functions on $S$, where $\mathcal{S}(\nu_i)$ denotes the class of integrable simple functions with respect to $\nu_i$, $i = 1, 2, \ldots, n$. If for a given $(n+1)$-tuple of real numbers $(p_1^{(k)}, p_2^{(k)}, \ldots, p_n^{(k)}, q^{(k)})$,* where $k = 1$ or $2$, $1 \leq p_i^{(k)} \leq \infty$, $i =$

*$1, 2, \ldots, n$, $1 \leq q^{(k)} \leq \infty$. $T$ is of types $(p_1^{(1)}, p_2^{(1)}, \ldots, p_n^{(1)}, q^{(1)})$ and $(p_1^{(2)}, p_2^{(2)}, \ldots, p_n^{(2)}, q^{(2)})$, that is,*

$$\|T(f_1, f_2, \ldots, f_n)\|_{q^{(k)}} \leq M_k \prod_{i=1}^{n} \|f_i\|_{p_i^{(k)}} \quad (k = 1, 2),$$

*where $f_i \in S(\nu_i)$. Then $T$ is of type $(p_1, p_2, \ldots, p_n, q)$ for*

$$\left.\begin{aligned} \frac{1}{p_i} &= \frac{1-t}{p_i^{(1)}} + \frac{t}{p_i^{(2)}}, \\ \frac{1}{q} &= \frac{1-t}{q^{(1)}} + \frac{t}{q^{(2)}}, \end{aligned}\right\}$$

*where $i = 1, 2, \ldots, n$; $0 < t < 1$, and the inequality*

$$\|T(f_1, f_2, \ldots, f_n)\|_q \leq M_1^{1-t} M_2^t \prod_{1=1}^{n} \|f_i\|_{p_i}$$

*holds.*

*Moreover, if all the $p_i$'s are finite, $T$ can be extended by continuity to $L^{p_1}(\nu_1) \times L^{p_2}(\nu_2) \times \cdots \times L^{p_n}(\nu_n)$, preserving the above inequality.*

## 1.2 Multipliers

Let $G$ be a locally compact abelian group. A bounded linear operator $T$ from $L^p(G)$ to $L^p(G)$, for $1 \leq p < \infty$, is said to be a Fourier multiplier operator if it commutes with translations, i.e., $\tau_x T = T\tau_x \ \forall x \in G$. It is well known that for each such $T$ there exists a bounded measurable function $\phi \in L^\infty(\hat{G})$ such that $(Tf)^\wedge = \phi\hat{f}, \ \forall f \in L^p \cap L^2(G)$. We denote by $M_p(\hat{G})$ the space of all such $\phi$'s. $M_p(\hat{G})$ is a Banach space with the norm

$$\|\phi\|_{M_p(\hat{G})} = \|T\|,$$

where $\|T\|$ denotes the operator norm of $T$ associated to $\phi$ as above. For general facts on multipliers see [31] or [39].

For $\phi \in M_p(\hat{G})$, $\|\phi\|_\infty \leq \|\phi\|_{M_p(\hat{G})}$. If $\{\phi_n\}$ is a sequence in $M_p(\hat{G})$ and $\phi_n \to \phi$ pointwise, then $\phi$ may not belong to $M_p(\hat{G})$. But we have the following result, whose proof is easy.

**Theorem 1.2** [29] *Let* $\{\phi_n\} \subseteq M_p(\hat{G})$ *and* $\phi_n \to \phi$ *a.e. with* $\sup_n \|\phi_n\|_{M_p(\hat{G})} < \infty$, *then* $\phi \in M_p(\hat{G})$.

The multiplier problem is to characterize $M_p(\hat{G})$. For $p = 1$ and $p = 2$ these spaces have been characterized [31]:

$$M_1(\hat{G}) \approx M(G)^\wedge \text{ and } M_2(\hat{G}) \approx L^\infty(\hat{G}).$$

Thus $M_1(\hat{G}) \approx (C_0(G))^*$ and $M_2(\hat{G}) \approx (L^1(\hat{G}))^*$, where $X^*$ denotes the dual of a Banach space $X$. For $1 < p < 2$ Figà-Talamanca and Gaudry characterized the multiplier spaces $M_p(\hat{G})$ as duals of certain Banach spaces. For $1 \le p < 2$, define the spaces $A_p(G)$, as follows $A_p(G) := \{f : f = \sum_n g_n * h_n$ such that $\{g_n\} \subseteq L^p(G), \{h_n\} \subseteq L^{p'}(G)$ with $\sum_n \|g_n\|_p \|h_n\|_{p'} < \infty\}$.

Then $A_p(G)$ is a Banach space with the norm

$$\|f\|_{A_p(G)} = \inf\left\{\sum_n \|g_n\|_p \|h_n\|_{p'} : f = \sum_n g_n * h_n\right\},$$

where the infimum is taken over all representations of $f$. We refer to [26], [31] for details.

**Theorem 1.3** $(A_p(G))^* \approx M_p(\hat{G})$.

In the above theorem, for $\phi \in M_p(\hat{G})$, the identification with the linear functional is given by

$$K_\phi F = \sum_n T_\phi g_n * h_n(0) \text{ where } F = \sum_n g_n * h_n,$$

and it can be proved that $K_\phi F$ is well-defined, i.e. is independent of the representation of $F$. Conversely, given $\psi \in A_p(G)^*$, the operator $T$ is defined by the dual action as

$$< Tf, g >= \psi(f * g) \quad f \in L^p,\ g \in L^{p'}.$$

Observe that $A_1(G) \approx C_0(G)$ and $A_2(G) \approx L^1(\hat{G})$.

In this thesis we restrict ourselves to $G = \mathbb{R}^N$ , $\mathbb{T}^N$ or $\mathbb{Z}^N$. We will study the relationship between $M_p(\hat{\mathbb{R}^N})$, $M_p(\mathbb{Z}^N)$, and $M_p(\mathbb{T}^N)$.

## 1.3 Restriction and Extension Problems

A natural question, called the restriction problem is : Suppose $\phi \in M_p(\hat{\mathbb{R}^N})$ is such that $\phi|_{\mathbb{Z}^N}$ makes sense, then is it true that $\phi|_{\mathbb{Z}^N} \in M_p(\mathbb{Z}^N)$? In 1965, de Leeuw [21], considered this problem for bounded measurable functions which are regulated , i.e., such that $\lim_{r\to 0} \frac{1}{|I_r(x)|} \int_{I_r(x)} \phi(t)dt = \phi(x)$, for all $x$, where $I_r(x)$ is the ball of radius $r$ and centre $x$. He proved that

**Proposition 1.1** *If $\phi \in M_p(\hat{\mathbb{R}^N})$ and is regulated then $\phi|_{\mathbb{Z}^N} \in M_p(\mathbb{Z}^N)$.*

In [18], this result is stated for a closely related class of functions $\phi$ called normalized. Since $\mathbb{R}/\mathbb{Z}$ can be identified with $\mathbb{T}$, another question that arises naturally is: If $\phi \in M_p(\hat{\mathbb{R}})$ and is one-periodic then does $\phi \in M_p(\mathbb{T})$? In [21] de Leeuw proved the following

**Theorem 1.4** *Let $\phi$ be a bounded measurable function on $\mathbb{T}$. Define $\psi$ on $\mathbb{R}$ by $\psi(x) = \phi(e^{2\pi i x})$. Then the following are equivalent.*
*(i) $\phi \in M_p(\mathbb{T})$*
*(ii) $\psi \in M_p(\hat{\mathbb{R}})$.*
*If (i) and (ii) hold then $||\phi||_{M_p(\mathbb{T})} = ||\psi||_{M_p(\hat{\mathbb{R}})}$.*

In the same paper, de Leeuw [21], proved various important results concerning the relation between $M_p(\hat{\mathbb{R}^N})$, $M_p(\mathbb{Z}^N)$, $M_p(\mathbb{T}^N)$, $M_p(\hat{\mathbb{R}}_b^N)$ where $\mathbb{R}_b^N$ is the Bohr compactification of $\mathbb{R}^N$. For general locally compact abelian groups the restriction problem translates as the Homomorphism theorem for multipliers [23], which states that "Let $G_1$ and $G_2$ be two locally compact abelian groups. If $\phi \in M_p(\hat{G}_1) \cap C(\hat{G}_1)$ and $\pi : \hat{G}_2 \longrightarrow \hat{G}_1$ is a continuous homomorphism then $\phi \circ \pi \in M_p(\hat{G}_2)$ and its multiplier norm does not exceed the multiplier norm of $\phi$."

## The Extension Problem, a Survey

In the reverse direction we can ask the following question: For $\phi$ in $M_p(\mathbb{Z})$ can we construct $\psi$ (continuous?) on $\mathbb{R}$ such that $\psi|_{\mathbb{Z}} = \phi$. The most natural extension is the piecewise constant function $\psi$ which agrees with $\phi$ at integer points, i.e., for $\phi \in M_p(\mathbb{Z})$ define $\psi(\xi) = \sum_n \phi(n)\chi_{[0,1)}(\xi - n)$. Jodeit [29] proved that $\psi \in M_p(\hat{\mathbb{R}})$ for $1 < p < \infty$ (Since $\psi$ is not continuous in general, $\psi$ cannot be in $M_1(\hat{\mathbb{R}})$). Another natural extension is the piecewise linear function $\psi$ such that $\psi(n) = \phi(n)$ for $\phi \in M_p(\mathbb{Z})$. In [29] Jodeit also proved the following:

Define $\Lambda(\xi) = \max(1 - |\xi|, 0)$. If $\phi \in M_p(\mathbb{Z})$ then $\psi(\xi) = \sum_n \phi(n)\Lambda(\xi - n)$ belongs to $M_p(\hat{\mathbb{R}})$ for $1 \leq p < \infty$.

Figà-Talamanca and Gaudry [25] considered the piecewise quadratic extension of $\phi \in M_p(\mathbb{Z})$. Let $\Lambda(\xi) = \max(1 - |\xi|, 0)$, define $\psi$ by $\psi(\xi) = \sum_n \phi(n)\Lambda^2(\xi - n)$ for $\phi \in M_p(\mathbb{Z})$. Then $\psi$ belongs to $M_p(\hat{\mathbb{R}})$ for $1 \leq p < \infty$. They used the characterization of $M_p(\hat{G})$ as the dual of the Banach space $A_p(G)$ to prove this result. All the above extensions can be generalized in the following manner. Let $\Lambda$ be a bounded measurable function on $\hat{\mathbb{R}}$ such that *supp* $\Lambda \subseteq [0, 1]$ and $\Lambda(0) = 1$. Define

$$\psi(\xi) = \sum_n \phi(n)\Lambda(\xi - n) \tag{1.1}$$

for $\phi \in M_p(\mathbb{Z})$. Then $\psi|_{\mathbb{Z}} = \phi$. The question is "does $\psi \in M_p(\hat{\mathbb{R}})$?"

However a more general problem is posed if we relax the condition on $\Lambda$ and construct $\psi$ as in Eqn.(1.1) (provided the sum exists) and ask whether $\psi \in M_p(\hat{\mathbb{R}})$ whenever $\phi \in M_p(\mathbb{Z})$. In this general setting $\psi$ need not be an extension of $\phi$. In particular if *supp* $\Lambda$ is compact, then $\psi$ is well defined by Eqn.(1.1) and under some additional conditions Asmar, Berkson, and Gillespie answered this question in the affirmative by adapting the duality technique of [25]. Their result was further improved by Berkson, Paluszyński, and Guido Weiss in [11] using a powerful transference method which is described in the next section. In [29], Jodeit

also considered a class of functions $\Lambda$ which do not necessarily have compact support. These function satisfy $\sum_n |\Lambda(\xi+n)| < \infty$ (in which case sum in Eqn.(1.1) is defined), and also that $\Lambda = \hat{F}$ for some $F \in L^1(\mathbb{R})$ and *supp* $F \subseteq [\frac{1}{4}, \frac{3}{4}]$ with $\sum_n |\hat{F}^{\#}(n)| < \infty$ , where $F^{\#}$ is the 1-periodic extension of $F$ from [0,1). For the general problem that we will investigate, let us define:

**Definition 1.1** *A bounded measurable function $\Lambda$ on $\mathbb{R}$ is said to be a summability kernel for $L^p(\mathbb{R})$ multipliers , for $1 \leq p < \infty$, if*

$$W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}} \phi(n)\Lambda(\xi - n) \tag{1.2}$$

*is defined pointwise a.e. and belongs to $M_p(\hat{\mathbb{R}})$ whenever $\phi \in M_p(\mathbb{Z})$ and $||W_{\phi,\Lambda}||_{M_p(\hat{\mathbb{R}})} \leq C_{p,\Lambda}||\phi||_{M_p(\mathbb{Z})}$, where $C_{p,\Lambda}$ is a constant depending only upon $p, \Lambda$.*

For $p = 1$, it is well known that $M_p(\hat{G}) \simeq M(G)$. So the 'extensions problem' deals with the 'extension' of measures from the circle group $\mathbb{T}$ to measures on the Real line $\mathbb{R}$ (i.e. through summability kernel). We have observed the following result which talks about the existence of an 'extended' measure in a much more general setup.

**Theorem 1.5** *Let $X$ and $Y$ be two locally compact Hausdorff space. Suppose $\phi : X \to Y$ is a continuous, open, and onto map. Then if $\nu \in M(Y)$ there exists $\mu \in M(X)$ such that for any $f \in C(Y)$*

$$\int_X f \circ \phi(x) d\mu(x) = \int_Y f(y) d\nu(y).$$

**Proof:** We will prove the result for $\nu \in M^+(Y)$, the space of positive regular bounded Borel measures on $Y$. For arbitrary $\nu \in M(Y)$ the result will follow from the Jordan decomposition of signed measures [16]. From the regularity and boundedness of $\nu \in M^+(Y)$ it is easy to see that there exists a sequence of pairwise disjoint compact subsets $K_n$ of $Y$ such that for

any $\nu$-measurable subset $A$ of $Y$

$$\nu(A) = \sum_{n=1}^{\infty} \nu(A \cap K_n).$$

Let $x \in \phi^{-1}(K_n)$, then there exists a neighbourhood $U_x$ of $x$ such that $\bar{U}_x$ is compact. Now $\{\phi(U_x) : x \in \phi^{-1}(K_n)\}$ is an open cover for $K_n$. So, we have a finite subcover $\{\phi(U_{x_i}) : i = 1, ..., m\}$ for $K_n$. Consider $E_n = \cup_{i=1}^{m} \bar{U}_{x_i} \cap \phi^{-1}(K_n)$. Hence, for each $n$, $E_n$ is compact and $\phi(E_n) = K_n$. Now define $\mathcal{U}_n = \{f \circ \phi_n : f \in C(K_n)\}$ where $\phi_n = \phi|_{E_n}$. Again it is easy to see that $\mathcal{U}_n$ is a closed linear subspace of $C(E_n)$. For each $n$ let us define a bounded linear functional $T_n : \mathcal{U}_n \to \mathbb{C}$ by

$$T_n(f \circ \phi_n) = \int_{K_n} f(y)d\nu(y) = \int_{Y} f(y)d\nu_n(y),$$

where $\nu_n(A) = \nu(A \cap K_n)$, for $\nu$-measurable subset $A$ of $Y$. Let $\tilde{T}_n$ be a Hahn-Banach extension of $T_n$ to all of $C(E_n)$. Then $||\tilde{T}_n|| = ||T_n|| = T_n(1) = \tilde{T}_n(1)$ where 1 is the constant function 1. Now we claim that $\tilde{T}_n$ is a positive linear functional for each $n$. Suppose not then there exists $f \in C(E_n)$ such that $f \geq 0$ and $\tilde{T}_n f < 0$. For this $f$ define $g = ||f||_\infty - f$. Then $0 \leq g \leq ||f||_\infty$. Now

$$\begin{aligned}
\tilde{T}_n g &= \tilde{T}_n(||f||_\infty - f) \\
&= ||f||_\infty \tilde{T}_n(1) - \tilde{T}_n(f) \\
&> ||f||_\infty ||\tilde{T}_n|| \qquad \text{(as } \tilde{\mathrm{T}}_\mathrm{n}\mathrm{f} < 0) \\
&\geq ||g||_\infty ||\tilde{T}_n||,
\end{aligned}$$

which is a contradiction. Hence by Riesz Representation Theorem [16] there exists a unique $\mu_n \in M(E_n)$ such that

$$\tilde{T}_n f = \int_{E_n} f d\mu_n \qquad \forall f \in C(E_n).$$

Therefore in particular

$$\int_{E_n} f \circ \phi_n(x) d\mu_n(x) = \int_{K_n} f(y) d\nu_n(y).$$

Define $\mu = \sum_{n=1}^{\infty} \mu_n$. Since $||\mu_n|| = ||\tilde{T}_n|| = ||\nu_n||$, we have $||\mu|| = \sum_{n=1}^{\infty} ||\mu_n|| = \sum_{n=1}^{\infty} ||\nu|| = ||\nu|| < \infty$. Then $\mu \in M^+(X)$. For $f \in C(Y)$ we have $\int_X f \circ \phi(x) d\mu(x) = \int_Y f(y) d\nu(y)$.

**Remark:** The above theorem is a generalization of the following result for locally compact abelian groups given in [28].

**Theorem 1.6** *Let $G$ be a locally compact abelian group and $G_0$ is a closed subgroup of $G$. Then for every positive definite function $p_0$ on $G_0$ there exists a positive definite function $p$ on $G$ such that $p|_{G_0} = p_0$.*

The proof of our theorem follows the same line of argument given in [28] for the above theorem. In the case of groups, $E_n$'s in the proof are constructed in the same fashion by translating a relatively compact neighbourhood of identity.

## 1.4 The Transference Method

This technique was first developed and used by Calderón [15] in 1968 where he proved inequalities for the ergodic Hilbert transform by transferring the corresponding inequalities for the ordinary Hilbert transform. These results for the Ergodic Hilbert transform were earlier proved by Cotlar [17] but his proofs were highly technical and complicated.

Subsequently Coifman and Weiss [18], [19] further developed Calderón's technique and showed that several operators could be viewed as transferred operators via suitable representations.

In the late eighties and early nineties this technique was used in the above mentioned "extension" problem of multipliers by Asmar, Berkson, Gillespie, Muhly, Paluszyński, Weiss, and others in a series of papers [1], [2], [11], [13]. We now describe the transference method (see [18] for proofs and further details).

Let $G$ be an *amenable* group which means $G$ is a locally compact group satisfying the

following property: Given a compact set $K \subset G$ and $\epsilon > 0$ there exists an open neighbourhood $V$ of identity having finite left (or right ) Haar measure $\lambda(V)$ such that $\frac{\lambda(VK^{-1})}{\lambda(V)} \leq 1+\epsilon$, where for a measurable subset $E$, $\lambda(E)$ is its Haar measure. Every locally compact abelian group is amenable [28]. Let $R$ be a uniformly bounded representation of $G$ on $L^p(\mathcal{M})$, where $\mathcal{M}$ is a $\sigma$- finite measure space. That is

(i) The map $u \mapsto R_u$ is strongly continuous

(ii) $R_{uv} = R_u R_v \;\; \forall u, v \in G$

(iii) $\sup_{u \in G} ||R_u|| = c < \infty$.

For $k \in L^1(G)$ define the operator $H_k$ on $L^p(\mathcal{M})$ as

$$H_k f(.) = \int_G k(u) R_{u^{-1}} f(.) du.$$

This operator $H_k$ is called the transferred operator. It is easy to see that $||H_k||_{L^p(\mathcal{M})} \leq ||k||_1 ||f||_{L^p(\mathcal{M})}$. The general transference result by Coifman and Weiss gives an improvement on this inequality. Note that the $L^p$- multiplier norm of convolution by $k$, $N_p(k)$ is less than $||k||_1$.

**Theorem 1.7** (Theorem 2.4, [18]) *The operator $H_k$ is a bounded operator from $L^p(\mathcal{M})$ to $L^p(\mathcal{M})$ with an operator norm not exceeding $c^2 N_p(k)$ where $N_p(k)$ is the operator norm of the convolution operator $f \mapsto k * f$ on $L^p(G)$.*

Guido Weiss, Soria, and Zaloznik [38] gave an alternative proof of the fact that $\chi_{[0,1)}$ is a summability kernel for $1 < p < \infty$ by using the transference technique. They defined a strongly continuous representation given by $(R_u f)^\wedge(\xi) = e^{2\pi i u[\xi]} \hat{f}(\xi)$ for $u \in \mathbb{T}$ and $f \in L^2 \cap L^p(\mathbb{R})$. Then for $k \in L^1(\mathbb{T})$, $(H_k f)^\wedge(\xi) = \sum_{n \in \mathbb{Z}} \hat{k}(n) \chi_{[0,1)}(\xi - n)$, where $H_k$ is the transferred operator on $L^p(\mathbb{R})$. The uniform boundedness of $\{R_u\}$ was achieved by a transference argument from $l_p$-boundedness of discrete Hilbert transform. For other summability

kernels we do not have such nice representations. However Berkson, Paluszyński, and Weiss [11] introduced transference couples, i.e., two families of strongly continuous operators, one of which is used to transfer the operators and the other compensates for the fact that the first family is not a representation. More specifically:

Let $(S,T) = (\{S_u\}, \{T_u\})$ $u \in G$, be a pair of strongly continuous maps of $G$ into the space of bounded operators on $L^p(\mathcal{M})$ such that

(i) $S_u T_v = T_{uv} \quad \forall u, v \in G$

(ii) $\sup_{u \in G} \|S_u\| = c_S < \infty$

(iii) $\sup_{u \in G} \|T_u\| = c_T < \infty$.

This pair $(S,T)$ is called a transference couple. The general transference couples result is

**Theorem 1.8** [11] *Let $(S,T)$ be a transference couple. The operator $H_k f = \int_G k(u) T_{u^{-1}} f(.) du$ for $k \in L^1(G)$ and $f \in L^p(\mathcal{M})$ is a bounded operator from $L^p(\mathcal{M})$ to $L^p(\mathcal{M})$ with operator norm not exceeding $c_S c_T N_p(k)$.*

In [11], for $\Lambda \in M_p(\hat{\mathbb{R}})$ with *supp* $\Lambda \subseteq [0,1)$, they defined the following transference couple $(\{S_u\}, \{T_u\})$

$$(S_u f)^\wedge(\xi) = \sum_{n \in \mathbb{Z}} e^{2\pi i u n} \chi_{[0,1)}(\xi - n) \hat{f}(\xi)$$

$$(T_u f)^\wedge(\xi) = \sum_{n \in \mathbb{Z}} e^{2\pi i u n} \Lambda(\xi - n) \hat{f}(\xi) \quad \text{for } f \in L^2 \cap L^p(\mathbb{R}).$$

By using the general transference couples result, cited above, they proved that $\Lambda$ is a summability kernel for $1 \leq p < \infty$. Further they showed that the condition *supp* $\Lambda \subset [0,1]$ can be relaxed to simply that *supp* $\Lambda$ be compact. In the same paper they also proved inequalities for maximal operators. This method of transference couples gives the corresponding results for Fourier integral once the results for Fourier series are known. As an example they derived the Carleson-Hunt theorem for real line, by transferring from the circle group.

Berkson, Gillespie, and Muhly [13] view multiplier operators on a locally compact group as being transferred operators, in the following sense. Let $G$ be a locally compact abelian group, $(\mathcal{M}, \mu)$ be an arbitrary measure space, and suppose $R$ is a strongly continuous uniformly bounded representation of $G$ on $L^p(\mathcal{M})$. Then by Stone's theorem [36] there exists a unique countably additive regular spectral measure $\mathcal{E}(.)$, defined on the Borel sets of $\hat{G}$ and acting in $L^2(\mu)$ such that

$$R_u = \int_{\hat{G}} \gamma(u) d\mathcal{E}(\gamma) \quad \text{for all } u \in G.$$

Let $\Theta : \hat{G} \longrightarrow \mathbb{C}$ be a bounded Borel measurable function. Define a bounded linear operator $\mathcal{T}_\Theta : L^2(\mathcal{M}) \longrightarrow L^2(\mathcal{M})$ by

$$\mathcal{T}_\Theta = \int_{\hat{G}} \Theta(\gamma) d\mathcal{E}(\gamma).$$

If $\mathcal{T}_\Theta$ satisfies a weak-type $(p,p)$ inequality for $f \in L^2 \cap L^p(\mathcal{M})$ then it has a unique extension from $L^2 \cap L^p(\mathcal{M})$ to a linear mapping $\mathcal{T}_\Theta^{(p)}$ from $L^p(\mathcal{M})$ to the space of complex valued measurable functions on $X$. If $\Theta = \hat{k}$ for some $k \in L^1(G)$ then

$$\mathcal{T}_{\hat{k}}^{(p)} f = H_k f \quad \forall f \in L^p(\mathcal{M}), \tag{1.3}$$

where $H_k$ is the transferred operator defined by

$$H_k f(.) = \int_G k(u) R_{u^{-1}} f(.) du \quad \forall f \in L^p(\mathcal{M}).$$

By considering $\mathcal{M} = G$ and the representation $R$ to be $R_u f(x) = f(ux)$ for $u, x \in G$ and $f \in L^p(G)$ one can see that if $\phi \in M_p(\hat{G})$ then $\mathcal{T}_\phi = T_\phi$, where $T_\phi$ is the multiplier operator corresponding to $\phi$.

The Coifman and Weiss transference theorem generalizes to

**Theorem 1.9** (Theorem 2.1[13]) *Suppose for $1 \leq p < \infty$ the following conditions hold*

*(i) for each $u \in G$, $R_u$ can be extended from $L^2(\mu) \cap L^p(\mu)$ to a continuous linear mapping $R_u^{(p)}$ of $L^p(\mathcal{M})$ into $L^p(\mathcal{M})$,*

*(ii)* $c_p = \sup\{\|R_u^{(p)}\| : u \in G\} < \infty$.

*Then* $u \mapsto R_u^{(p)}$ *is a strongly continuous representation of* $G$ *on* $L^p(\mathcal{M})$ *and, for each* $\phi \in M_p(\hat{G}) \cap C(\hat{G})$, *the operator* $\int_{\hat{G}} \phi d\mathcal{E}$ *extends from* $L^2(\mathcal{M}) \cap L^p(\mathcal{M})$ *to a bounded linear mapping of* $L^p(\mathcal{M})$ *into* $L^p(\mathcal{M})$ *with norm not exceeding* $c_p^2 \|\phi\|_{M_p(\hat{G})}$.

Berkson and Gillespie in [14] used the transference technique to transfer any $L^p(\mathbb{T})$ multiplier to a bounded linear operator on a closed subspace $X$ of $L^p(\mathcal{M})$ for $1 < p < \infty$, where $\mathcal{M}$ is an arbitrary measure space. Suppose $z \mapsto R_z$ is a strongly continuous representation of the circle group $\mathbb{T}$ on $X$. Then in the following theorem this representation is decomposed in terms of projection operators.

**Theorem 1.10** (Theorem 1.1 [14]) *Let* $X, \mu, R$ *be as above, and suppose* $1 < p < \infty$. *Then there is a unique sequence of idempotent operators* $\{P_n\}_{n=-\infty}^{\infty} \subseteq \mathcal{B}(X)$ *such that*

$$P_n P_m = 0 \quad for\ m \neq n$$

*and (the series converges in the strong operator topology of* $\mathcal{B}(X)$*)*

$$R_z = \sum_{n=0}^{\infty} z^n P_n + \sum_{n=1}^{\infty} z^n P_n \quad for\ z \in \mathbb{T}.$$

*This unique sequence* $\{P_n\}_{n=-\infty}^{\infty}$ *is given by*

$$P_n x = \int_{\mathbb{T}} z^{-n} R_z x dz \quad for\ x \in X,\ n \in \mathbb{Z}.$$

*Corresponding to each* $\phi \in M_p(\mathbb{Z})$, *the operator* $\mathcal{T}_\phi \in \mathcal{B}(X)$ *such that*

$$\mathcal{T}_\phi = \sum_{n \in \mathbb{Z}^N} \phi(n) P_n$$

*Each series on the right converges in the strong operator topology of* $\mathcal{B}(X)$. *Moreover the operator norm of* $\mathcal{T}_\phi$ *satisfies*

$$\|\mathcal{T}_\phi\| \leq c^2 \|\phi\|_{M_p(\mathbb{Z})}$$

*where* $c = \sup\{\|R_z\| : z \in \mathbb{T}\}$.

By considering $X = L^p(\mathbb{R})$ and $P_n f = T_{\chi_{[n,n+1)}} f$, one gets that $\chi_{[0,1)}$ is a summability kernel for $L^p(\mathbb{R})$ multipliers for $1 < p < \infty$. In this same paper, for $1 \leq p < \infty$, they modified the method of Jodeit [29] suitably to extend certain class of functions in $M_p(\mathbb{Z})$ to continuous functions belonging to $M_p(\hat{\mathbb{R}})$. To state their result we need the following definition.

**Definition 1.2** *The support of $T_\phi$, denoted by supp $T_\phi$, where $T_\phi$ is the multiplier transform of $L^2(G)$ of $\phi \in L^\infty(\hat{G})$, is the closed subset of $G$ given by*

$$supp\ T_\phi = -\Sigma(\phi)$$

*where $\Sigma(\phi)$ is the spectrum of $\phi$ as a bounded measurable function on $\hat{G}$* ([28], (40.2)).

**Theorem 1.11** *Let $\phi \in L^\infty(\mathbb{R})$ and suppose that supp $T_\phi \subseteq [\frac{1}{4}, \frac{3}{4}]$ . Then there is a unique bounded continuous function $\phi_c$ on $\mathbb{R}$ such that $\phi_c = \phi$ almost everywhere on $\mathbb{R}$. Suppose further that $1 \leq p < \infty$. Then $\phi \in M_p(\hat{\mathbb{R}})$ if and only if $\phi_c|_{\mathbb{Z}} \in M_p(\mathbb{Z})$. If this is the case then*

$$||\phi_c|_{\mathbb{Z}}||_{M_p(\mathbb{Z})} \leq ||\phi||_{M_p(\hat{\mathbb{R}})} \leq 2^{\frac{1}{p^*}} ||\phi_c|_{\mathbb{Z}}||_{M_p(\mathbb{Z})},$$

*where $p^* = \max(p, p')$.*

## 1.5 Description of Thesis

Recall the definition of a summability kernel

**Definition 1.3** *A bounded measurable function $\Lambda$ on $\mathbb{R}$ is said to be a summability kernel for $L^p(\mathbb{R})$ multipliers , for $1 \leq p < \infty$, if*

$$W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}} \phi(n)\Lambda(\xi - n) \tag{1.4}$$

*is defined pointwise a.e. and belongs to $M_p(\hat{\mathbb{R}})$ whenever $\phi \in M_p(\mathbb{Z})$ and $||W_{\phi,\Lambda}||_{M_p(\hat{\mathbb{R}})} \leq C_{p,\Lambda}||\phi||_{M_p(\mathbb{Z})}$ where $C_{p,\Lambda}$ is a constant depending only upon $p, \Lambda$.*

Let us denote $S_p(\mathbb{R})$ to be the set of all summability kernels for $L^p(\mathbb{R})$ multipliers. It is natural to study the structure of this space $S_p(\mathbb{R})$. In [12] Berkson, Paluszyński, and Weiss used wavelets to prove that the space is dense in $M_p(\hat{\mathbb{R}})$ in the weak* topology. It still remains to characterize the space $S_p(\mathbb{R})$. In Chapter 2, we study summability kernels for $L^p(\mathbb{R})$ multipliers. We first characterize summability kernels for $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ multipliers. As mentioned earlier, all previous classes of summability kernels either have compact support or are Fourier transforms of a compactly supported integrable functions. In §2 we prove fairly general results which make precise the reasons why summability kernels allow the transference of multipliers from $\mathbb{T}$ to $\mathbb{R}$. In §3 we restrict ourselves to $p = 1$. In this section we give various types of summability kernels which transfer discrete measures, continuous measures, absolutely continuous measures (with respect to Lebesgue measure) on $\mathbb{T}$ to measures on $\mathbb{R}$ with the same property.

In Chapter 3 we study a somewhat different kind of extension. Instead of assuming $\phi$ to be a multiplier we assume $\phi$ to be an arbitrary sequence in $l_p(\mathbb{Z})$. In §2 of this chapter we show that "If $S \in L^1(\mathbb{R})$ and *supp* $S \subseteq [\frac{1}{4}, \frac{3}{4}]$ with $\sum_{n\in\mathbb{Z}} |\hat{S}(n)|^p < \infty$ then for $\phi \in l_{p'}$ we have

$$W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}}) \text{ for } \begin{cases} q \in [\frac{2p}{3p-2}, \frac{2p}{2-p}] & \text{if } 1 < p < 2 \\ q \in [\frac{2p}{p+2}, \frac{2p}{p-2}] & \text{if } 2 < p < \infty. \end{cases}$$

For $p = 2$ from the standard result $L^2 * L^2(\hat{G}) = (L^1(G))^\wedge$, one gets $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$ for $1 \leq q < \infty$. Further we prove that if $\phi \in l_p(\mathbb{Z})$ and $S \in L^p(\mathbb{R})$ then $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$ for $1 \leq q < \infty$. In §3 we have proved a maximal counterpart of our result.

Before describing the contents of Chapter 4 we need some notation and concepts.

$L^{(p,q)}$ **Spaces:** Let $(\mathcal{M}, \mu)$ be a $\sigma$-finite measure space.

**Definition 1.4** [39] *The distribution function $\lambda_f$ of a function $f$ on $\mathcal{M}$ is defined as*

$$\lambda_f(t) = \mu\{x \in \mathcal{M} : |f(x)| > t\} \quad (t \geq 0)$$

For $f \in L^p(\mathcal{M})$ it is easy to see that $t^p\lambda_f(t) \leq \|f\|_p^p$ for $t > 0$ and in fact $\|f\|_p^p = p\int_0^\infty t^{p-1}\lambda_f(t)dt$.

**Definition 1.5** *Suppose $f$ is a measurable function on $\mathcal{M}$. The decreasing rearrangement of $f$ is the function $f^*$ defined on $[0,\infty)$ by*

$$f^*(t) = \inf\{s : \lambda_f(s) \leq t,\ t \geq 0)\}$$

It is not hard to see that if $f \in L^p(\mathcal{M})$ then $\|f\|_p^p = \int_0^\infty f^*(t)^p dt$.

For $1 \leq p, q \leq \infty$ we define the space $L^{p,q}(\mathcal{M})$ as the space of all measurable functions $f$ satisfying

$\|f\|_{pq}^* = (\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t})^{\frac{1}{q}} < \infty$ when $1 \leq p < \infty, 1 \leq q < \infty$

and

$\|f\|_{p\infty}^* = \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) < \infty$ when $1 \leq p \leq \infty$.

Clearly $L^{p,p} = L^p$. For $1 < p \leq \infty, 1 \leq q \leq \infty$, $L^{p,q}$ is a Banach space with the following norm

$$\|f\|_{pq} = (\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}} m(t)]^q \frac{dt}{t})^{\frac{1}{q}}$$

where $m(t) = \frac{1}{t}\int_0^t f^*(u)du$. $\|\ \|_{pq}^*$ is not a norm, for $1 < p \leq \infty$, but the following inequalities hold.

**Theorem 1.12** *If $f \in L^{p,q}(\mathcal{M}), 1 < p \leq \infty$ then*

$$\|f\|_{pq}^* \leq \|f\|_{pq} \leq \frac{p}{p-1}\|f\|_{pq}^*.$$

**Definition 1.6** *Let $G$ be a locally compact abelian group with Haar measure $\mu$. An operator $T$ on $L^p(G)$ is said to be of weak-type $(p,p)$ if*

$$\lambda_{Tf}(t) \leq \left(\frac{C}{t}\|f\|_p\right)^p \tag{1.5}$$

*for $t > 0$ and for all $f \in L^p(G)$.*

In Chapter 4 we study the translation invariant operators on $L^p(\mathbb{R})$ which are not necessarily bounded on $L^p$ for $1 \leq p < \infty$ but satisfy weak $(p,p)$ inequalities. Asmar, Berkson and Gillespie [8] proved that for each such operator there exists a $\phi \in L^\infty(\hat{G})$ such that $(Tf)^\wedge = \phi\hat{f}$ for $f \in L^2 \cap L^p(G)$. The functions $\phi$ associated to these weak type translation invariant operators are called weak type $(p,p)$ multipliers. So it is natural to establish an appropriate parallel between Fourier multipliers (which can be called strong type multipliers) and weak type multipliers. Thus we can ask same questions pertaining to extensions and restrictions of weak type multipliers between $\mathbb{R}$ and $\mathbb{T}$. Though in this thesis we are concentrating only on the problem of extensions, we felt the need of making a survey of the restriction problem for weak type multipliers for the sake of perspective. This is the content of the first two sections of this chapter. Major contributions in this area are the work of Asmar, Berkson, and Gillespie [8], [3], [4], [6], [7], [9].

In the next four sections of this chapter we study the extension problems concerning weak type multipliers. The extension by $\Lambda = \max(1-|x|,0)$ and $\Lambda = \hat{S}$ where $S \in L^1(\mathbb{R})$ and *supp* $S \subseteq [\frac{1}{4}, \frac{3}{4}]$ with $\sum_{n\in\mathbb{Z}} |(S^\#)^\wedge(n)| < \infty$ (originally proved for strong type multipliers by Jodeit [29]) has been proved for weak type multipliers by Asmar, Berkson, and Gillespie. We will prove the weak type counterpart of Berkson, Paluszyński, and Weiss's result [11] i.e. "if $\Lambda \in M_p(\hat{\mathbb{R}})$ for $1 < p < \infty$ and *supp* $\Lambda \subseteq [\frac{1}{4}, \frac{3}{4}]$ then $\Lambda$ is a weak type summability kernel. In §5 we relax the condition that *supp* $\Lambda \subset [\frac{1}{4}, \frac{3}{4}]$ . In §6 as an application of this result we prove a weak type analogue of de Leeuw's extension theorem [39].

# Chapter 2

# Summability Kernels For $L^p$ Multipliers

## 2.1 Introduction

In this chapter we study summability kernels which transfer $L^p(\mathbb{T})$ multipliers to $L^p(\mathbb{R})$ multipliers. For results regarding summability kernels we refer to [29], [1], [11], and [25].

Let us consider the following set:

$S_p^0(\mathbb{R}) = \{\Lambda \in L^\infty(\hat{\mathbb{R}}) :$ For each finitely supported $\phi \in M_p(\mathbb{Z})$, the function $W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}} \phi(n)\Lambda(\xi - n)$ belongs to $M_p(\hat{\mathbb{R}})$ and there exists a constant $C_{p,\Lambda}$ such that $\|W_{\phi,\Lambda}\|_{M_p(\hat{\mathbb{R}})} \leq C_{p,\Lambda}\|\phi\|_{M_p(\mathbb{Z})}\}$.

**Lemma 2.1** *(i) For $1 < p < \infty$, let $\Lambda \in S_p^0(\mathbb{R})$. If for $\phi \in M_p(\mathbb{Z})$, the series*

$$\sum_{n\in\mathbb{Z}} \phi(n)\Lambda(\xi - n) = W_{\phi,\Lambda}(\xi)$$

*converges a.e., then $W_{\phi,\Lambda} \in M_p(\hat{\mathbb{R}})$.*

*(ii) If $p = 1$, $\Lambda \in S_1^0(\mathbb{R})$ and $\delta_\Lambda = \sup_\xi \sum_{n\in\mathbb{Z}} |\Lambda(\xi + n)| < \infty$ then $W_{\phi,\Lambda} \in M_1(\hat{\mathbb{R}})$ for every $\phi \in M_1(\mathbb{Z})$.*

**Proof:**

(i) For a.e. $\xi$, we have

$$W_{\phi,\Lambda}(\xi) = \lim_{N\to\infty} \sum_{-N}^{N} \phi(n)\Lambda(\xi - n) = \lim_{N\to\infty} W_{\phi_N,\Lambda}(\xi)$$

where $\phi_N(n) = \begin{cases} \phi(n) & \text{if } |n| \leq N \\ 0 & \text{otherwise.} \end{cases}$

Since $1 < p < \infty$, $\phi_N \in M_p(\mathbb{Z})$, and $||\phi_N||_{M_p(\mathbb{Z})} \leq C_p||\phi||_{M_p(\mathbb{Z})}$, where $C_p$ is a constant independent of $N$. This follows from F & M.Riesz theorem [23]. Since $\Lambda \in S_p^0(\mathbb{R})$ we have

$$\begin{aligned} ||W_{\phi_N,\Lambda}||_{M_p(\hat{\mathbb{R}})} &\leq C_{p,\Lambda}||\phi_N||_{M_p(\mathbb{Z})} \\ &\leq C_pC_{p,\Lambda}||\phi||_{M_p(\mathbb{Z})}. \end{aligned}$$

Hence $W_{\phi_N,\Lambda}(\xi) \to W_{\phi,\Lambda}(\xi)$ pointwise a.e. and boundedly , so $W_{\phi,\Lambda} \in M_p(\hat{\mathbb{R}})$.

(ii) For $p = 1$, the additional condition guarantees the convergence of $\sum_{n\in\mathbb{Z}} \phi(n)\Lambda(\xi - n)$ for every $\phi \in M_1(\mathbb{Z})$. If $K_N$ is the $N$th Fejér kernel, let $\phi_N = \hat{K}_N\phi$. Then again $W_{\phi_N,\Lambda}(\xi) \to W_{\phi,\Lambda}(\xi)$ pointwise a.e. and boundedly.

This lemma suggests the following definition :

**Definition 2.1** *Let* $S_p(\mathbb{R}) = \{\Lambda \in L^\infty(\hat{\mathbb{R}}) :$ For every $\phi \in M_p(\mathbb{Z})$, the series $\sum_{n\in\mathbb{Z}} \phi(n)\Lambda(\xi - n)$ converges a.e. to a function $W_{\phi,\Lambda} \in M_p(\hat{\mathbb{R}})$ and there exists a constant $C_{p,\Lambda}$ such that $||W_{\phi,\Lambda}||_{M_p(\hat{\mathbb{R}})} \leq C_{p,\Lambda}||\phi||_{M_p(\mathbb{Z})}\}$.

$S_p(\mathbb{R})$ *is said to be the set of summability kernels.*

Clearly $S_p(\mathbb{R}) \subseteq S_p^0(\mathbb{R})$. In §2 of this chapter we characterize the space $S_p^0(\mathbb{R})$ for $p = 1$ and for $p = 2$. As mentioned in the introduction, Jodeit in his paper [29], proved the following results regarding summability kernels.

(a) The indicator function $\chi_{[0,1)}$, of the interval $[0,1)$, belongs to $S_p(\mathbb{R})$ for $1 < p < \infty$

(b) The function $\Lambda(\xi) = \max\,(1 - |\xi|, 0)$ belongs to $S_p(\mathbb{R})$ for $1 \leq p < \infty$.

(c) If $S \in L^1(\mathbb{R})$ is such that $suppS \subseteq [\frac{1}{4}, \frac{3}{4}]$ and if the 1-periodic extension of $S$ has absolutely summable Fourier series, then $\hat{S} \in S_p(\mathbb{R})$ for $1 \leq p < \infty$.

In 1971, Figà-Talamanca and Gaudry [25] proved that if $\Lambda$ is the triangular function as defined in (b) then $\Lambda^2$ is also a summability kernel. They used the duality relation $A_p(\mathbb{R})^* \simeq M_p(\hat{\mathbb{R}})$. By introducing the concept of transference couples Berkson, Paluszynski and Weiss in [11] proved that if $\Lambda \in M_p(\hat{\mathbb{R}})$ and has compact support then $\Lambda$ is a summability kernel. Following the methods of Figà-Talamanca and Gaudry, Asmar, Berkson, and Gillespie in [1] gave a large class of summability kernels, namely the class $\mathcal{F} = \{J \in L^1(\mathbb{R}) :$ $J$ is continuous, $\hat{J}$ has compact support, $\hat{J}$ is absolutely continuous, $\hat{J}' \in L^2(\mathbb{R})\}$. It turns out that if $J \in \mathcal{F}$, then both $J$ and $\hat{J}$ are summability kernels. In §2 we prove fairly general results which make precise the reasons why summability kernels allow the transference of multipliers from $\mathbb{T}$ to $\mathbb{R}$. In all previously known classes, either the summability kernel $\Lambda$ has compact support or $\Lambda = \hat{f}$ for some $f \in L^1(\mathbb{R})$ with compact support. Theorem 1.1 provides classes of summability kernels which do not satisfy either of these two conditions. However, we are unable to give a complete characterization.

In §3 we restrict ourselves to the case $p = 1$. We investigate some properties of measures which are stable under transference by summability kernels, such as the properties of being discrete, continuous or absolutely continuous.

## 2.2 Summability Kernels for $L^p(\mathbb{R}), 1 < p < \infty$

For a function $\Lambda \in L^\infty(\mathbb{R})$ denote $\delta_\Lambda = \operatorname{ess\,sup}_{\xi} \sum_{n\in\mathbb{Z}} |\Lambda(\xi + n)|$ . In the following proposition we characterize $S_p^0(\mathbb{R})$, for $p = 1$ and $p = 2$.

**Proposition 2.1** *(i)* $S_2^0 = S_2 = \{\Lambda \in L^\infty(\hat{\mathbb{R}}) : \delta_\Lambda < \infty\}$.

*(ii)* $S_1^0 = \{\Lambda \in L^1(\mathbb{R})^\wedge : \Lambda = \hat{F} \text{ with } \delta_F < \infty\}$.

**Proof:**

(i) Let $\Lambda \in L^\infty(\hat{\mathbb{R}})$ and suppose $\delta_\Lambda < \infty$. Then clearly $\Lambda \in S_2(\mathbb{R})$. Let $\Lambda \in S_2^0(\mathbb{R})$. Consider any finite sequence $\{\phi(n)\}$ such that $|\phi(n)| = 1$ for all $n$. Then $\phi \in M_2(\mathbb{Z})$ and $||\phi||_{M_2(\mathbb{Z})} = 1$. We may assume that $\Lambda$ is defined everywhere. For $\xi_0 \in \mathbb{R}$, let $\phi_{\xi_0}(n) = \text{sgn } \Lambda(\xi_0 - n)$. Then $||\phi_{\xi_0}||_{M_2(\mathbb{Z})} = 1$ and we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |\Lambda(\xi_0 - n)| &= \sup_N \sum_{|n|\leq N} |\Lambda(\xi_0 - n)| \\
&= \sup_N \sum_{|n|\leq N} \phi_{\xi_0}(n)\Lambda(\xi_0 - n) \\
&\leq \sup_N || \sum_{|n|\leq N} \phi_.(n)\Lambda(. - n)||_\infty \\
&\leq C_{2,\Lambda}||\Lambda||_{M_2(\hat{\mathbb{R}})} < \infty \qquad\qquad (\text{as } \Lambda \in S_2^0(\mathbb{R})).
\end{aligned}$$

So $\delta_\Lambda < \infty$. This implies that $\Lambda \in S_2(\mathbb{R})$. We already know that $S_2(\mathbb{R}) \subseteq S_2^0(\mathbb{R})$. Hence $S_2(\mathbb{R}) = S_2^0(\mathbb{R})$.

(ii) Let $\Lambda = \hat{F}$, where $F \in L^1(\mathbb{R})$ and $\delta_F < \infty$. For a finite sequence $\{\phi(n)\}$ let $P(x) = \sum_n \phi(n)e^{2\pi inx}$. Then

$$\begin{aligned}
W_{\phi,\Lambda}(\xi) &= \sum_n \phi(n)\Lambda(\xi - n) \\
&= \sum_{n\in\mathbb{Z}} \phi(n)\hat{F}(\xi - n) \\
&= \sum_n \phi(n) \int_{\mathbb{R}} F(x)e^{-2\pi ix(\xi-n)}dx \\
&= \int_{\mathbb{R}} F(x)\sum_n \phi(n)e^{2\pi ixn}e^{-2\pi ix\xi}dx \\
&= \int_{\mathbb{R}} F(x)P^{\#}(x)e^{-2\pi ix\xi}dx,
\end{aligned}$$

where $P^{\#}$ is the 1-periodic extension of $P$. So,

$$W_{\phi,\Lambda}(\xi) = (FP^{\#})^{\wedge}(\xi). \tag{2.1}$$

We have $FP^{\#} \in L^1(\mathbb{R})$ as $P^{\#} \in L^{\infty}(\mathbb{R})$. Thus

$$\begin{aligned} \|W_{\phi,\Lambda}\|_{M_1(\hat{\mathbb{R}})} &\leq \|FP^{\#}\|_1 \\ &= \int_0^1 \sum_n |F(x+n)||P(x)|dx \\ &\leq \delta_F \|P\|_{L^1(\mathbb{T})} = \delta_F \|\phi\|_{M_1(\mathbb{Z})}. \end{aligned}$$

Hence $\Lambda \in S_1^0(\mathbb{R})$.

Conversely, suppose $\Lambda \in S_1^0(\mathbb{R})$. Then taking $\phi(n) = \delta_{n,0}$ we have $\Lambda \in M_1(\hat{\mathbb{R}}) = M(\mathbb{R})^{\wedge}$. So $\Lambda = \hat{\mu}$ for some $\mu \in M(\mathbb{R})$. For a finite sequence $\{\phi(n)\}$ and $P$ as above we have

$$\begin{aligned} W_{\phi,\Lambda}(\xi) &= \sum_n \phi(n)\Lambda(\xi - n) \\ &= \sum_{n\in\mathbb{Z}} \phi(n)\hat{\mu}(\xi - n) \\ &= \sum_n \phi(n) \int_{\mathbb{R}} e^{-2\pi i x(\xi-n)} d\mu(x) \\ &= \int_{\mathbb{R}} \sum_n \phi(n) e^{2\pi i x n} e^{-2\pi i x \xi} d\mu(x) \\ &= \int_{\mathbb{R}} e^{-2\pi i x \xi} P^{\#}(x) d\mu(x) \\ &= (P^{\#}\mu)^{\wedge}(\xi), \end{aligned}$$

where $P^{\#}\mu$ denotes the measure given by $d(P^{\#}\mu)(x) = P^{\#}(x)d\mu(x)$ . Then

$$\|P^{\#}\mu\|_{M(\mathbb{R})} = \|W_{\phi,\Lambda}\|_{M_1(\hat{\mathbb{R}})} \leq C_{1,\Lambda}\|\phi\|_{M_1(\mathbb{Z})} = C_{1,\Lambda}\|P\|_{L^1(\mathbb{T})}.$$

For $k \in \mathbb{Z}$ let $\mu_k = \tau_{-k}(\mu|_{[k,k+1)})$, i.e., $\mu_k$ is a measure supported on $[0, 1)$. Then $\|P\mu_k\|_{M(\mathbb{T})} \leq \|P^{\#}\mu\|_{M(\mathbb{R})} \leq C_{1,\Lambda}\|P\|_{L^1(\mathbb{T})}$. Since trigonometric polynomials are dense

in $C[0,1)$, for every continuous function $f$ on $[0,1)$, we have $||f\mu_k||_{M(\mathbb{T})} \leq C_{1,\Lambda}||f||_{L^1(\mathbb{T})}$. Let '$m$' denote the Lebesgue measure on $\mathbb{T}$, and let $E$ be a Lebesgue measurable set with $m(E) = 0$. Then for a given $\epsilon > 0$ consider a continuous function $f \neq 0$ on $\mathbb{T}$ such that $f(x) = 1$ on $E$ and $||f||_1 < \epsilon$. Then $\mu_k(E) < \epsilon$. So $\mu_k << m$. This is true for every $k$, hence $\mu$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. So there exists $F \in L^1(\mathbb{R})$ such that $d\mu = Fdx$, so that $\Lambda = \hat{F}$. Now

$$\begin{aligned} ||W_{\phi,\Lambda}||_{M_1(\hat{\mathbb{R}})} = ||P^{\#}F||_{L^1(\mathbb{R})} &= \sum_k \int_0^1 |P(x)|F(x+k)|dx \\ &= \int_0^1 |P(x)||F|^{\#}(x)dx. \end{aligned}$$

As $\Lambda \in S_1^0(\mathbb{R})$, we have

$$\begin{aligned} \int_0^1 |P(x)||F|^{\#}(x)dx &\leq C_{1,\Lambda}||\phi||_{M_1(\mathbb{Z})}||\Lambda||_{M_1(\mathbb{R})} \\ &= C_{1,\Lambda}||P||_1||\Lambda||_{M_1(\mathbb{R})} \qquad (as\ \phi(n) = \hat{P}(n)). \end{aligned}$$

Therefore $|F|^{\#}$ defines a continuous linear functional on $L^1(\mathbb{T})$, so by duality $|F|^{\#} \in L^\infty(\mathbb{T})$, and $|||F|^{\#}||_{L^\infty(\mathbb{T})} = \delta_F < \infty$.

**Remarks:** The two conditions appearing in Proposition 2.1 for $p = 1$ and for $p = 2$ seem to be very different. We will analyse these further to obtain a more unified formulation.

1. Let $p = 2$. The condition on $\Lambda$, namely $\delta_\Lambda < \infty$ is equivalent to saying that for a.e. $\xi \in \mathbb{R}$, the sequences $\{\Lambda(\xi+n)\}_{n\in\mathbb{Z}} \in l_1(\mathbb{Z})$. In other words, these sequences define, by convolution, multiplier operators on $l_1(\mathbb{Z})$. Now if $\Lambda = \hat{F}$ where $F$ is a tempered distribution (note that if $\delta_\Lambda < \infty$, $\Lambda \in L^1(\hat{\mathbb{R}})$, so $F$ can be defined as a function), then $(F_\xi^{\#})^\wedge(n) = \Lambda(\xi + n)$ a.e. $\xi \in \hat{\mathbb{R}}$, where $F_\xi = e^{2\pi i\xi\cdot}F$. Hence the condition $\delta_\Lambda < \infty$ can be reformulated as

($a_2$) for a.e. $\xi \in \hat{\mathbb{R}}$, $F_\xi \in M_1(\mathbb{T})$ with (essentially) uniformly bounded operator norms.

($b_2$) for a.e. $x$, $\Lambda_x^{\#} \in L^\infty(\mathbb{T}) = M_2(\mathbb{T})$, where $\Lambda_x = e^{2\pi ix\cdot}\Lambda$.

Let the notation be as in Remark 1.

2. Let $p = 1$. Now the condition $\delta_F < \infty$ is equivalent to saying that for a.e. $x \in \mathbb{R}$, the sequences $\{F(x+n)\}_{n\in\mathbb{Z}}$ define, by convolution, operators on $l^1(\mathbb{Z})$ or in other words, we have

(b$_1$) $\Lambda_x^\# \in M_1(\mathbb{T})$ ( where $(F(x+.))^\wedge = \Lambda_x^\#$).

Further $\delta_F < \infty$ implies that

(a$_1$) $F_\xi^\# \in l^\infty(\mathbb{Z}) = M_2(\mathbb{T})$.

From (b$_2$) and (b$_1$) above we get a condition which we will show is necessary for $\Lambda$ to be a summability kernel for $L^p(\mathbb{R})$ multipliers. For this, we need to define the following.

Let $\mathcal{F}_p = \{\Lambda \in L^\infty(\hat{\mathbb{R}}) : \text{for a.e. } x \in [0,1),\ \Lambda_x^\# \in M_p(\mathbb{T}) \text{ with } ||\Lambda_x^\#||_{M_p(\mathbb{T})} \in L^\infty[0,1)\}$.

**Proposition 2.2** $S_p(\mathbb{R}) \subseteq \mathcal{F}_p$ for $1 \le p \le 2$.

**Proof:** Let $\Lambda \in S_p(\mathbb{R})$, then $\Lambda_x \in S_p(\mathbb{R})$. In fact

$$\begin{aligned} W_{\phi,\Lambda_x}(\xi) &= \sum_n \phi(n)\Lambda_x(\xi-n) \\ &= e^{2\pi i x\xi}\sum_n \phi(n)e^{-2\pi i x n}\Lambda(\xi-n). \end{aligned}$$

Since $\phi_x(n) = e^{-2\pi i x n}\phi(n)$ belongs to $M_p(\mathbb{Z})$ whenever $\phi \in M_p(\mathbb{Z})$ with equal norm, we have $W_{\phi,\Lambda_x} \in M_p(\hat{\mathbb{R}})$ and

$$\begin{aligned} ||W_{\phi,\Lambda_x}||_{M_p(\hat{\mathbb{R}})} &= ||W_{\phi_x,\Lambda}||_{M_p(\hat{\mathbb{R}})} \\ &\le C_{p,\Lambda}||\phi||_{M_p(\mathbb{Z})}. \end{aligned}$$

Now take $\phi(n) \equiv 1$. Then $W_{1,\Lambda_x}(\xi) = \sum_n e^{-2\pi i x(\xi-n)}\Lambda(\xi-n)$ belongs to $M_p(\hat{\mathbb{R}})$ and is 1-periodic. Thus by de Leeuw's result [21], $W_{1,\Lambda_x} \in M_p(\mathbb{T})$ and $||W_{1,\Lambda_x}||_{M_p(\mathbb{T})} \le C_{p,\Lambda}$ for all $x \in [0,1)$. Therefore $||(e^{2\pi i x\cdot}\Lambda)^\#||_{M_p(\mathbb{T})} \in L^\infty[0,1)$.

In [1], Asmar, Berkson, and Gillespie introduced the class $\mathcal{F}$ (mentioned earlier in this thesis). By suitably refining their proof (Theorem III.4 [1]), we have the following theorem, which improves on their result. For the sake of completeness we give here a detailed sketch of the proof of our theorem.

**Theorem 2.1** *(i) If $\Lambda_1,\ \Lambda_2 \in \mathcal{F}_p \cap A(\mathbb{R})$ then $\Lambda = \Lambda_1\Lambda_2 \in S_p^0(\mathbb{R})$.*

*(ii) If, in addition, either $\delta_{\Lambda_1} < \infty$ or $\delta_{\Lambda_2} < \infty$ then $\Lambda \in S_p(\mathbb{R})$.*

**Sketch of the Proof:** Let $\hat{F}_j = \Lambda_j$ where $F_j \in L^1(\mathbb{R})$ for $j = 1, 2$. Define the map $U$ on $A_p(\mathbb{R})$ as follows:

For $h_1,\ h_2 \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz class functions,

$$U(h_1 * h_2)(x) = \sum_n F(x+n) h_1 * h_2(x+n) \quad \text{for a.e. } x \in [0,1)$$

where $F = F_1 * F_2$. We will show that $\|U(h_1 * h_2)\|_{A_p(\mathbb{T})} \leq C\|h_1\|_p\|h_2\|_{p'}$, where $C$ is a constant depending on $F_1$ and $F_2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. By standard approximation one can then show that for $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$, $\|U(f*g)\|_{A_p(\mathbb{T})} \leq C\|f\|_p\|g\|_{p'}$. Now we extend $U$ to whole of $A_p(\mathbb{R})$ in the following way. Let $h \in A_p(\mathbb{R})$ with $h = \sum_{n\in\mathbb{Z}} f_n * g_n$, where $f_n \in L^p(\mathbb{R})$ and $g_n \in L^{p'}(\mathbb{R})$ for all $n$. Define $h_N = \sum_{n=1}^{N} f_n * g_n$. Then by the above inequality $\{Uh_N\}$ is Cauchy in $A_p(\mathbb{T})$. Let $Uh_N$ converges to $H$ in $A_p(\mathbb{T})$ as $N$ tends to infinity. We define $Uh = H$.

For $t \in \mathbb{R}$ define $\Phi_t : \mathbb{T} \to \mathbb{C}$ and $\Psi_t : \mathbb{T} \to \mathbb{C}$ for a.e. $x \in [0,1)$ by

$$\Phi_t(x) = \sum_n F_1(x+t+n) h_1(x+n)$$

and

$$\Psi_t(x) = \sum_n F_2(x+t+n) h_2(x+n).$$

Now consider the representation $R$ of $\mathbb{Z}$ acting on $L^p(\mathbb{R})$ by $n \mapsto R_n$ defined by $R_n f(x) = f(x-n)$. $\{F_1(x+n)\}_{n\in\mathbb{Z}}$ belongs to $l_1(\mathbb{Z})$ for almost every $x$. Hence the transferred operator

corresponding to this sequence and the representation $R$, defined by

$$H_{F_{1_x}} h(t) = \sum_n F_1(x+n)h(t-n)$$

for a.e $x \in [0,1)$, is a bounded linear operator on $L^p(\mathbb{R})$ and the norm is bounded by $C_p N_p(F_{1_x})$. Now

$$\begin{aligned}\int_{\mathbb{R}} ||\Phi_t||^p_{L^p(\mathbb{T})} dt &= \int_{\mathbb{R}}\int_{\mathbb{T}} |\sum_n F_1(x+t+n)h_1(x+n)|^p dx dt \\ &= \int_{\mathbb{T}}\int_{\mathbb{R}} |\sum_n F_1(x+n)h_1(x-t+n)|^p dt dx.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}} ||\Phi_t||^p_{L^p(\mathbb{T})} dt \leq C_p \int_{\mathbb{T}} ||\Lambda^{\#}_{1x}||_{M_p(\mathbb{T})} ||h_1||_{L^p(\mathbb{R})} dx. \tag{2.2}$$

Similarly,

$$\int_{\mathbb{R}} ||\Psi_t||^p_{L^p(\mathbb{T})} dt \leq C_p \int_{\mathbb{T}} ||\Lambda^{\#}_{2x}||_{M_p(\mathbb{T})} ||h_2||_{L^p(\mathbb{R})} dx. \tag{2.3}$$

As in [1], it is easy to see that $U(h_1 * h_2)(x) = \int_{\mathbb{R}} \Phi_t * \Psi_t(x) dx$ for a.e. $x \in \mathbb{T}$. So,

$$||U(h_1 * h_2||_{A_p(\mathbb{T})} \leq C_{p,\Lambda_1,\Lambda_2} ||h_1||_p ||h_2||_{p'}. \tag{2.4}$$

For $\{\phi(n)\}$ a finitely supported sequence, let $P(x) = \sum_n \phi(n) e^{2\pi i n x}$. As $W_{\phi,\Lambda} \in M_p(\hat{\mathbb{R}})$, we have

$$K_{W_{\phi,\Lambda}}(h_1 * h_2) = T_{W_{\phi,\Lambda}} h_1 * h_2(0),$$

where $K_{W_{\phi,\Lambda}}$ is the linear functional on $A_p(\mathbb{R})$ corresponding to $W_{\phi,\Lambda} \in M_p(\hat{\mathbb{R}})$. Also $W_{\phi,\Lambda} = (P^{\#}F)^{\wedge}$ (by Eq. 2.1 ). Thus

$$\begin{aligned}K_{W_{\phi,\Lambda}}(h_1 * h_2) &= P^{\#}F * h_1 * h_2(0) \\ &= \int_{\mathbb{R}} P^{\#}F(x) h_1 * h_2(-x) dx \\ &= \int_0^1 P(x) \sum_{n\in\mathbb{Z}} F(x+n) \tilde{h_1} * \tilde{h_2}(x+n) dx,\end{aligned}$$

where $\tilde{f}(x) = f(-x)$. If $K_\phi$ is the linear functional on $A_p(\mathbb{T})$ corresponding to $\phi \in M_p(\mathbb{Z})$, then for $F = \sum_{n\in\mathbb{Z}} f_n * g_n$ in $A_p(\mathbb{T})$, we have

$$\begin{aligned} K_\phi(F) &= \sum_{n\in\mathbb{Z}} T_\phi f_n * g_n(0) \\ &= \sum_{n\in\mathbb{Z}} P * f_n * g_n(0) \\ &= \sum_{n\in\mathbb{Z}} \int_0^1 P(x) f_n * g_n(-x)dx \\ &= \int_0^1 P(x)\tilde{F}(x)dx. \end{aligned}$$

Therefore $K_{W_{\phi,\Lambda}}(h_1 * h_2) = K_\phi(U(h_1 * \tilde{h_2}))$. Hence,

$$\begin{aligned} |K_{W_{\phi,\Lambda}}(h_1 * h_2)| &\le \|\phi\|_{M_p(\mathbb{Z})} \|U(h_1 * h_2)\|_{A_p(\mathbb{T})} \\ &\le C_{p,\Lambda_1,\Lambda_2} \|\phi\|_{M_p(\mathbb{Z})}. \end{aligned}$$

So $\Lambda \in S_p^0(\mathbb{R})$. Now if either $\delta_{\Lambda_1}$ or $\delta_{\Lambda_2}$ is finite then $\delta_\Lambda < \infty$. Then by Lemma 2.1, $\Lambda \in S_p(\mathbb{R})$.

**Corollary 2.1** *If $\Lambda = e^{-x^2}$ then $\Lambda \in S_p(\mathbb{R})$.*

**Proof:** $\Lambda = e^{-x^2} = e^{\frac{-x^2}{2}} e^{\frac{-x^2}{2}}$. Let us denote $\Lambda_1 = e^{\frac{-x^2}{2}}$ and $F_1(x) = \hat{\Lambda}_1(x) = e^{-2\pi^2 x^2}$. Therefore

$$\begin{aligned} \|\Lambda^{\#}_{1_x}\|_{M_p(\mathbb{T})} &\le \|F_1(x + .)\|_{l_1(\mathbb{Z})} \\ &= \sum_{n\in\mathbb{Z}} e^{-2\pi^2(x+n)^2} \\ &\le e^{-2\pi^2x^2} \left( \sum_{n\ge 0} e^{-2\pi^2 n^2} + \sum_{n<0} e^{-2\pi^2 n^2} e^{-2\pi^2 n} \right). \end{aligned}$$

So, $\sup_x \|\Lambda^{\#}_{1_x}\|_{M_p(\mathbb{T})} < \infty$. Hence $\Lambda \in S_p(\mathbb{R})$.

From the above corollary it is clear that there is a large class of summability kernels which do not have compact support nor are Fourier transforms of compactly supported integrable functions. We can also generate such summability kernels, for $1 < p < \infty$, by using Stečkin's theorem [23].

**Theorem 2.2** (Stečkin) *Let $\psi$ be a function of bounded variation on $\hat{\mathbb{R}}$ and $1 < p < \infty$. Then $\phi \in M_p(\hat{\mathbb{R}})$ and there exists a constant $C_p$ such that*

$$\|\psi\|_{M_p(\hat{\mathbb{R}})} \leq C_p \ \max \ (|\psi(0)|, var\psi),$$

*where var $\psi$ is the total variation of $\psi$.*

Let $\Lambda \in M_p(\hat{\mathbb{R}})$ and $\Lambda_k = \chi_{[k,k+1)}\Lambda$. Then $W_{\phi,\Lambda} = \sum_k W_{\phi,\Lambda_k}$. We also have $\Lambda_k \in S_p(\mathbb{R})$ [11], and $\|W_{\phi,\Lambda_k}\|_{M_p(\mathbb{R})} \leq C\|\phi\|_{M_p(\mathbb{Z})}\|\Lambda_k\|_{M_p(\mathbb{R})}$. If $\sum_k \|\Lambda_k\|_{M_p(\hat{\mathbb{R}})} < \infty$ then $W_{\phi,\Lambda} \in M_p(\hat{\mathbb{R}})$. So if we choose $\Lambda$ such that $\|\Lambda_k\|_{M_p(\hat{\mathbb{R}})} = var\Lambda_k$ and $\sum_k var\Lambda_k < \infty$. Then by Stečkin's theorem we have $\Lambda \in S_p(\mathbb{R})$. In particular, we have the following proposition. We give the proof in detail for the sake of completeness.

**Proposition 2.3** *Suppose $\Lambda \in L^\infty(\hat{\mathbb{R}})$ is a differentiable function satisfying*

*(i)* $|\Lambda(\xi)| < \frac{C}{(1+|\xi|)^{1+\delta}}$ *and*

*(ii)* $|\Lambda'(\xi)| \leq \frac{C}{(1+|\xi|)^{1+\delta}}$, *for some constant $C$ and $\delta > 0$.*

*Then if $\phi \in M_p(\mathbb{Z})$, the function $W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}} \phi(n)\Lambda(\xi - n)$ is defined and belongs to $M_p(\hat{\mathbb{R}})$ for $1 < p < \infty$ and there exists a constant $C_{p,\Lambda}$ such that*

$$\|W_{\phi,\Lambda}\|_{M_p(\hat{\mathbb{R}})} \leq C_{p,\Lambda,}\|\phi\|_{M_p(\mathbb{Z})}.$$

**Proof:** Condition (ii) implies that $\Lambda$ is a function of bounded variation, hence $\Lambda \in M_p(\hat{\mathbb{R}})$ for $1 < p < \infty$. For each $k \in \mathbb{Z}$, and $1 < p < \infty$, $\Lambda_k \in M_p(\hat{\mathbb{R}})$. But *supp* $\Lambda_k \subseteq [k, k+1)$, hence $\Lambda_k$ is a summability kernel [11], and so $W_{\phi,\Lambda_k} \in M_p(\hat{\mathbb{R}})$ whenever $\phi \in M_p(\mathbb{Z})$. Moreover,

$$\begin{aligned} W_{\phi,\Lambda_k}(\xi) &= \sum_{n\in\mathbb{Z}} \phi(n)\Lambda_k(\xi - n) \\ &= \sum_{n\in\mathbb{Z}} \phi(n)\chi_k(\xi - n)\Lambda(\xi - n) \\ &= \tau_k(\sum_{n\in\mathbb{Z}} \phi(n)\chi_o(. - n)\tau_{-k}\Lambda(. - n))(\xi). \end{aligned}$$

Hence,

$$\begin{aligned}||W_{\phi,\Lambda_k}||_{M_p(\hat{\mathbb{R}})} &= ||W_{\phi,\chi_0\tau_{-k}\Lambda}||_{M_p(\hat{\mathbb{R}})} \\ &\leq C_p||\phi||_{M_p(\mathbb{Z})}||\,\chi_0\tau_{-k}\Lambda||_{M_p(\hat{\mathbb{R}})} && (by\ [11]) \\ &= C_{p,\Lambda}||\phi||_{M_p(\mathbb{Z})}||\,\Lambda_k||_{M_p(\hat{\mathbb{R}})}.\end{aligned}$$

But from (i) and (ii) we have

$$var\ \Lambda_k \leq \frac{C}{(1+|k|)^{1+\delta}}$$

and so using Stečkin's theorem we have

$$\begin{aligned}||W_{\phi,\Lambda}||_{M_p(\hat{\mathbb{R}})} &\leq \sum_{k\in\mathbb{Z}}||W_{\phi,\Lambda_k}||_{M_p(\hat{\mathbb{R}})} \\ &\leq C_{p,\Lambda}||\phi||_{M_p(\mathbb{Z})}\sum_k||\Lambda_k||_{M_p(\hat{\mathbb{R}})} \\ &\leq C_{p,\Lambda,\delta}\ ||\phi||_{M_p(\mathbb{Z})}.\end{aligned}$$

## 2.3 Continuous, Absolutely continuous, and discrete measures

For the case $p = 1$, $M_1(\hat{\mathbb{R}})$ and $M_1(\mathbb{Z})$ are identified with $M(\mathbb{R})$ and $M(\mathbb{T})$ respectively. So if $\phi \in M_1(\mathbb{Z})$ then $\phi = \hat{\nu}$ for some $\nu \in M(\mathbb{T})$, and if $\Lambda$ is a summability kernel we have $W_{\phi,\Lambda} \in M_1(\hat{\mathbb{R}})$. Thus $W_{\phi,\Lambda} = \hat{\mu}$ for some $\mu \in M(\mathbb{R})$. Here we study some properties of measures which are carried over from $\nu$ to $\mu$. For this we need to define the following set: $\mathcal{F}_0 = \{\Lambda \in S_1^0(\mathbb{R}) : \delta_\Lambda < \infty\}$.

**Theorem 2.3** *Let $\Lambda \in \mathcal{F}_0$, $\nu \in M(\mathbb{T})$ and define $\hat{\mu}(\xi) = W_{\hat{\nu},\Lambda}(\xi) = \sum_n \hat{\nu}(n)\Lambda(\xi - n)$, (here $\mu \in M(\mathbb{R})$).*

*(a) If $\nu$ is an absolutely continuous measure on $\mathbb{T}$, then $\mu$ is an absolutely continuous measure on $\mathbb{R}$ (both with respect to the Lebesgue measure).*

*(b) If $\nu$ is a discrete measure, then either $\mu \equiv 0$ or $\mu$ is a discrete measure.*

**Proof:**

(a) First assume that $d\nu(x) = P(x)dx$, where $P$ is a trigonometric polynomial. Then

$$\begin{aligned}\hat{\mu}(\xi) &= \sum_n \hat{P}(n)\Lambda(\xi - n) \\ &= \sum_n \hat{P}(n)\ \left(e^{2\pi in\cdot} g\right)^\wedge(\xi),\end{aligned}$$

where $\Lambda = \hat{g}$, $g \in L^1(\mathbb{R})$. Let

$$h(x) = \sum_n \hat{P}(n)\ e^{2\pi inx} g(x) = P^{\#}(x)g(x), \tag{2.5}$$

where $P^{\#}$ is the periodic extension of $P$. Since $P^{\#}$ is a bounded function, $h \in L^1(\mathbb{R})$ and $\hat{h} \equiv \hat{\mu}$, so

$$||h||_{L^1(\mathbb{R})} = ||\hat{\mu}||_{M_1(\hat{\mathbb{R}})} \leq C_{1,\Lambda}||\hat{\nu}||_{M_1(\mathbb{Z})} = C_{1,\Lambda}||P||_{L^1(\mathbb{T})}. \tag{2.6}$$

Now if $\nu$ is an absolutely continuous measure on $\mathbb{T}$, let $\hat{\nu} = \hat{F}$ for $F \in L^1(\mathbb{T})$. There exists a sequence $\{P_N\}$ of trigonometric polynomials such that $P_N \to F$ in $L^1(\mathbb{T})$. For each $P_N$ define $h_N$ as in Eqn. (2.5). Then from Eqn. (2.6)

$$||h_N - h_M||_{L^1(\mathbb{R})} \leq C_{1,\Lambda}||P_N - P_M||_{L^1(\mathbb{T})}.$$

Let $h_N \to h$ in $L^1(\mathbb{R})$. Now

$$\begin{aligned}|\hat{h}_N(\xi) - \hat{\mu}(\xi)| &\leq \sum_n |\hat{P}_N(n) - \hat{F}(n)|\ |\Lambda(\xi - n)| \\ &\leq ||P_N - F||_{L^1(\mathbb{R})}\ \delta_\Lambda.\end{aligned}$$

So, $\hat{h} = \hat{\mu}$. Hence, $d\mu(x) = h(x)dx$.

(b) Let $\nu = \sum_{j=1}^{\infty} \alpha_j \delta_{x_j}$ be a discrete measure on $\mathbb{T}$, $x_j \in \mathbb{T}$ and $\sum_j |\alpha_j| < \infty$. If $\Lambda = \hat{g}$, with $g \in L^1(\mathbb{R})$, (without loss of generality we may assume $g$ is continuous since $\Lambda \in L^1(\hat{\mathbb{R}})$) and $\tilde{g}(x) = g(-x)$, then

$$
\begin{aligned}
\hat{\mu}(\xi) &= \sum_n \hat{\nu}(n)\Lambda(\xi - n) \\
&= \sum_n \left(\sum_j \alpha_j e^{2\pi i n x_j}\right) \left(e^{2\pi i \xi \cdot}\, \tilde{g}\right)^\wedge(n) \\
&= \sum_j \alpha_j \sum_{n\in\mathbb{Z}} \left(\tau_{-x_j}\, e^{2\pi i \xi \cdot} \tilde{g}\right)(n) \\
&= \sum_j \alpha_j \sum_{n\in\mathbb{Z}} e^{2\pi i \xi (n+x_j)}\, \tilde{g}(x_j + n).
\end{aligned}
$$

The last but one equality follows by Poisson summation formula, since $\Lambda \in S_1^0(\mathbb{R})$ will imply $\sum\limits_n |g(x+n)| < \infty$ and $\Lambda \in \mathcal{F}_0$ will imply $\sum\limits_{n\in\mathbb{Z}} |\hat{g}(\xi+n)| < \infty$. Now it is clear that

$$\mu = \sum_{n\in\mathbb{Z}} \sum_j \alpha_j \tilde{g}(x_j + n)\ \delta_{x_j+n}.$$

Hence $\mu$ is either the zero measure or is discrete.

To consider similar results for continuous measures, we need some additional conditions on the summability kernel, and this is the content of the following two results, both of which use Wiener's lemma.

**Lemma 2.2** (Wiener's Lemma, [30]) *Let $\mu \in M(\mathbb{R})$. Then*

*(i)* $\mu(\{y\}) = \lim\limits_{\lambda\to\infty} \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} \hat{\mu}(\xi) e^{2\pi i y \xi} d\xi.$

*(ii)* $\sum |\mu(\{x\})|^2 = \lim\limits_{\lambda\to\infty} \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} |\hat{\mu}(\xi)|^2 d\xi.$

*In particular, a necessary and sufficient condition for $\mu$ to be continuous measure is that*

$$\lim_{\lambda\to\infty} \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} |\hat{\mu}(\xi)|^2 d\xi = 0.$$

**Theorem 2.4** *Let $\Lambda \in S_1(\hat{\mathbb{R}})$, and suppose that supp $\Lambda$ is compact. Then if $\nu$ is a continuous measure, so is $\mu$.*

**Proof:** By Wiener's Lemma

$$
\begin{aligned}
\mu\{y\} &= \lim_{\lambda\to\infty} \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} \hat{\mu}(\xi)\, e^{2\pi i \xi y}\, d\xi \\
&= \lim_{\lambda\to\infty} \frac{1}{2\lambda} \sum_{n\in\mathbb{Z}} \hat{\nu}(n) \int_{-\lambda}^{\lambda} \Lambda(\xi - n)\, e^{2\pi i \xi y}\, d\xi \\
&= \lim_{\lambda\to\infty} \frac{1}{2\lambda} \sum_{n} \hat{\nu}(n)\, I_\lambda^n(y),
\end{aligned}
$$

where

$$
I_\lambda^n(y) = \int_{-\lambda}^{\lambda} \Lambda(\xi - n)\, e^{2\pi i \xi y}\, d\xi.
$$

Let *supp* $\Lambda \subseteq [-N, N]$. Then for each $\lambda > 0$, if $|n| > N + \lambda$, clearly $I_\lambda^n(y) = 0\ \forall y \in \mathbb{R}$. Now let $\lambda > 2N$.

**Case 1** Suppose $|n| \le \lambda - N$, then

$$
\begin{aligned}
I_\lambda^n(y) &= \int_{-\lambda+n}^{\lambda+n} \Lambda(\xi)\, e^{2\pi i(\xi+n)y}\, d\xi \\
&= \hat{\Lambda}(-y)\, e^{2\pi i y n}
\end{aligned}
$$

since $[-N, N] \subset [-\lambda + n, \lambda + n]$.

**Case 2** $\lambda - N \le |n| \le \lambda + N$. Then

$$
I_\lambda^n(y) = \int_{-N+n}^{\lambda} \Lambda(\xi - n)\, e^{2\pi i \xi y}\, d\xi.
$$

So, in both the cases

$$
|I_\lambda^n(y)| \le \|\Lambda\|_{L^1(\hat{\mathbb{R}})}.
$$

Hence, for $\lambda > 2N$

$$
\begin{aligned}
\mu\{y\} &= \lim_{\lambda\to\infty} \frac{1}{2\lambda} \left( \sum_{|n|\le\lambda-N} \hat{\nu}(n)\, I_\lambda^n(y) + \sum_{N+\lambda\ge|n|\ge\lambda-N} \hat{\nu}(n) I_\lambda^n(y) \right) \\
&= \lim_{\lambda\to\infty} \frac{1}{2\lambda} \sum_{|n|\le[\lambda-N]} \hat{\nu}(n)\, e^{2\pi i y n}\, \hat{\Lambda}(-y) \\
&\quad + \lim_{\lambda\to\infty} \frac{1}{2\lambda} \sum_{N+\lambda\ge|n|>\lambda-N} \hat{\Lambda}(n)\, I_\lambda^n(y).
\end{aligned}
$$

The second limit is zero since the terms are bounded and the number of terms is at most $2N$. Applying Wiener's lemma for the continuous measure $\nu$ on $\mathbb{T}$ for the first limit we have

$$\mu\{y\} = \nu\{y_0\}\hat{\Lambda}(-y) = 0 \;\text{ where }\; y_0 \in [0,1) \; s.t. \; y = y_0 + 2\pi l \;\text{ for some }\; l \in \mathbb{Z}.$$

The hypothesis that *supp* $\Lambda$ be compact may be too restrictive. It can be replaced by the existence of a suitable decreasing radial $L^1$ - majorant $\Lambda_1$, i. e., a function $\Lambda_1$ satisfying

(a) $\Lambda_1$ is decreasing and radial

(b) $\Lambda_1 \in L^1(\mathbb{R})$

(c) $|\Lambda(\xi)| \leq |\Lambda_1(|\xi|)|$.

**Theorem 2.5** *Suppose $\Lambda \in \mathcal{F}_0$ and that $\Lambda$ has a decreasing radial $L^1$ - majorant $\Lambda_1$. Then $\mu$ is a continuous measure if $\nu$ is.*

**Proof:** Once again, we use Wiener's lemma.

Let

$$\begin{aligned}
I_\lambda &= \frac{1}{2\lambda}\int_{-\lambda}^{\lambda} |\hat{\mu}(\xi)|^2 \, d\xi \\
&= \frac{1}{2\lambda}\int_{-\lambda}^{\lambda} |\sum_n \hat{\nu}(n)\Lambda(\xi - n)|^2 \, d\xi \\
&\leq \frac{1}{2\lambda}\int_{-\lambda}^{\lambda} \left(\sum_n |\hat{\nu}(n)|^2 |\Lambda(\xi - n)|\right)\left(\sum_n |\Lambda(\xi - n)|\right) d\xi \\
&\leq \delta_\Lambda \left[\frac{1}{2\lambda}\sum_{|n|\leq 2\lambda} |\hat{\nu}(n)|^2 \int_{-\lambda}^{\lambda} |\Lambda(\xi - n)| d\xi \right.\\
&\qquad \left. + \frac{1}{2\lambda}\sum_{|n|>2\lambda} |\hat{\nu}(n)|^2 \int_{-\lambda}^{\lambda} |\Lambda(\xi - n)| d\xi\right] \\
&= \delta_\Lambda(I_1 + I_2), \text{ say.}
\end{aligned}$$

Now,

$$
\begin{aligned}
I_2 &\le \frac{1}{2\lambda} \sum_{|n|>2\lambda} |\hat{\nu}(n)|^2 \int_{-\lambda}^{\lambda} \Lambda_1(|\xi - n|)\, d\xi \\
&\le \frac{1}{2\lambda} \|\hat{\nu}\|_\infty \left( \sum_{n>2\lambda} \Lambda_1(|\lambda - n|) + \sum_{n<-2\lambda} \Lambda_1(|\lambda - n|) \right) \\
&\to 0 \quad \text{as } \lambda \to \infty,
\end{aligned}
$$

since $\sup\limits_{\xi} \sum\limits_{n} |\Lambda_1(\xi + n)| < \infty$ and $\Lambda_1$ is decreasing. Hence, using Wiener's lemma for $\mathbb{T}$ we get

$$
\begin{aligned}
\lim_{\lambda\to\infty} I_\lambda &\le \delta_\Lambda \lim_{\lambda\to\infty} \frac{1}{2\lambda} \sum_{|n|\le[2\lambda]} |\hat{\nu}(n)|^2 \int_{-\lambda}^{\lambda} |\Lambda(\xi - n)|\, d\xi \\
&\le \delta_\Lambda \|\Lambda\|_{L^1(\hat{\mathbb{R}})} \lim_{\lambda\to\infty} \sum_{|n|\le[2\lambda]} |\hat{\nu}(n)|^2 \\
&= 0.
\end{aligned}
$$

**Concluding Remarks:**

1. We do not know whether $S_p(\mathbb{R}) = S_p^0(\mathbb{R})$, except when $p = 2$. For $1 < p < 2$ in view of Proposition 2.1 we need suitable strong operator inequalities to prove that if $\Lambda \in S_p^0$, then for all $\phi \in M_p(\mathbb{Z})$, the series $\sum\limits_{n\in\mathbb{Z}} \phi(n)\Lambda(\xi - n)$ converges a.e.

2. In our attempt to give a characterization of $S_p(\mathbb{R})$ and $S_p^0(\mathbb{R})$, we have found a necessary condition (Proposition 2.2). The remarks preceeding Proposition 2.2, in fact also suggest another condition which generalizes ($a_1$) and ($a_2$). That this is also necessary remains a conjecture.

# Chapter 3

# Extensions Of Sequences To $L^p$ Multipliers

## 3.1 Introduction

In this chapter we study a different kind of extension. In the earlier chapter and in the existing literature the emphasis has been on the extensions of $L^p(\mathbb{T})$-multipliers to $L^p(\mathbb{R})$-multipliers. Here we will show that for some values of $p$ and $q$, every sequence in $l_p(\mathbb{Z})$ can be extended to an $L^q(\mathbb{R})$ multiplier by means of suitable summability kernels. The idea of our extension comes from the following result of Jodiet [29].

**Theorem 3.1** *Let $S \in L^1(\mathbb{R})$, supp $S \subseteq [\frac{1}{4}, \frac{3}{4}]$ and suppose its 1-periodic extension $S^\#$ from* $[0, 1)$ *has an absolutely summable Fourier series. Then*

$$W_{\phi,\hat{S}}(\xi) = \sum_{n\in\mathbb{Z}} \phi(n)\hat{S}(\xi - n) \tag{3.1}$$

*is in $M_p(\hat{\mathbb{R}})$ whenever $\phi \in M_p(\mathbb{Z})$ and its norm is bounded by $C_p\tau\|\phi\|_{M_p(\mathbb{Z})}$, where $\tau = \sum_n |(S^\#)^\wedge(n)|$ and $C_p$ is a constant which depends only on $p$.*

It is natural to ask what happens if we assume $(S^{\#})^\wedge \in l_p(\mathbb{Z})$ for $1 < p < \infty$. In this case, it follows from Lemma 3.1 in §3.2 and Hölder's inequality that the above sum converges for every sequence $\{\phi(n)\} \in l_{p'}(\mathbb{Z})$ where $\frac{1}{p} + \frac{1}{p'} = 1$, and defines a function $W_{\phi,\hat{S}}$ in $L^\infty(\hat{\mathbb{R}})$.

**Theorem 3.2** *Let $S \in L^1(\mathbb{R})$, supp $S \subseteq [\frac{1}{4}, \frac{3}{4}]$ and $\sum_n |(S^{\#})^\wedge(n)|^p < \infty$ for $1 < p < \infty$. Define $W_{\phi,\hat{S}}(\xi) = \sum_{n\in\mathbb{Z}} \phi(n)\hat{S}(\xi - n)$ for $\phi \in l_{p'}(\mathbb{Z})$. Then*

$$W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}}) \ \ for \begin{cases} q \in [\frac{2p}{3p-2}, \frac{2p}{2-p}] \ \ if\ 1 < p < 2 \\ q \in [\frac{2p}{p+2}, \frac{2p}{p-2}] \ \ if\ 2 < p < \infty. \end{cases}$$

*For $p = 2$, $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$ for $1 \le q < \infty$. Moreover,*

$$||W_{\phi,\hat{S}}||_{M_q(\hat{\mathbb{R}})} \le C\tau_p ||\phi||_p,$$

*where $\tau_p = (\sum_n |(S^{\#})^\wedge(n)|^p)^{\frac{1}{p}}$ and $C$ is a constant which depends only on $p$.*

By putting a further restriction on $S$ we will get $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$ for $1 \le q < \infty$ whenever $\phi \in l_p(\mathbb{Z})$, for $1 < p < 2$. In §3.3 we prove a maximal inequality.

## 3.2 The Main Theorem and Related Results

For the proof of Theorem 3.2, we first prove a lemma.

**Lemma 3.1** *Let $S \in L^1(\mathbb{R})$, supp $S \subseteq [\frac{1}{4}, \frac{3}{4}]$ and $\sum_n |(S^{\#})^\wedge(n)|^p < \infty$ for $1 \le p < \infty$. Then $\sup_\xi \sum_n |\hat{S}(\xi + n)|^p < \infty$.*
*Moreover for $p = 1$, $\sum_{n\in\mathbb{Z}} \hat{S}(\xi + n) = C$ for all $\xi \in \mathbb{R}$, where $C$ is a constant.*

**Proof:** Let $\rho \in C_c^\infty(\mathbb{R})$ be such that $\rho(x) = 1$ on $[\frac{1}{4}, \frac{3}{4}]$, *supp* $\rho \subseteq [\frac{1}{8}, \frac{7}{8}]$ and $|\rho(x)| \le 1 \ \forall x \in \mathbb{R}$. For $\xi \in [0, 1]$, define $h_\xi(x) = e^{-2\pi i\xi x}\rho(x)$. Then $h_\xi \in C_c^\infty(\mathbb{R})$ and

$$({h_\xi}^{(2)})^\wedge(y) = (2\pi i y)^2 \hat{h}_\xi(y).$$

Hence

$$\begin{aligned}|\hat{h}_\xi(y)| &= \frac{|\hat{h}^{(2)}_\xi(y)|}{|2\pi y|^2} \qquad (\text{for } y \neq 0)\\ &\leq \frac{||h^{(2)}_\xi||_1}{|2\pi y|^2} \leq \frac{C}{|y|^2},\end{aligned}$$

where the constant $C$ is independent of $\xi$. So in particular $|\hat{h}_\xi(n)| \leq \frac{C}{n^2}$ for $n \neq 0$. Now define $g_\xi(x) = h_\xi(x)S(x)$. Then *supp* $g_\xi \subseteq [\frac{1}{4}, \frac{3}{4}]$ and

$$\hat{g^{\#}_\xi}(n) = \hat{g}_\xi(n) = \int_{\frac{1}{4}}^{\frac{3}{4}} e^{2\pi i\xi x} S(x) e^{-2\pi i\xi n} dx = \hat{S}(\xi + n),$$

where $g^{\#}_\xi$ is the 1-periodic extension of $g_\xi$ given by $\sum_{n\in\mathbb{Z}} g_\xi(x+n)$. Also $\hat{g^{\#}_\xi}(n) = \hat{h^{\#}_\xi} * (S^{\#})^\wedge(n)$. Since $\hat{h^{\#}_\xi} \in l_1(\mathbb{Z})$ and $(S^{\#})^\wedge \in l_p(\mathbb{Z})$, it follows that $\hat{g^{\#}_\xi} \in l_p(\mathbb{Z})$ for $1 \leq p < \infty$, and

$$\sum_n |\hat{S}(\xi + n)|^p = \sum_n |\hat{g^{\#}_\xi}(n)|^p \leq ||\hat{h^{\#}_\xi}||_{l_1}^{\ p} ||(S^{\#})^\wedge||_{l_p}^{\ p} \leq C||(S^{\#})^\wedge||_p^p,$$

where the constant $C$ does not depend upon $\xi$. So, $\sup_{\xi\in[0,1]} \sum_n |\hat{S}(\xi + n)|^p < \infty$.

For $p = 1$, by the Fourier inversion, we may assume that $S$ is continuous. Now for a fixed $x$ define $g_x = e^{-2\pi ix\cdot}S$. Then $g_x$ is continuous and *supp* $g_x \subseteq [\frac{1}{4}, \frac{3}{4}]$ . Also $\hat{g}^{\#}_x(n) = \hat{S}(x + n)$, where $g^{\#}_x$ is the 1-periodic extension of $g_x$ from $[0, 1)$. Therefore $g^{\#}_x(t) = \sum_{n\in\mathbb{Z}} \hat{S}(x + n)e^{2\pi int}$ for $t \in [0, 1)$. As both sides of this equality are continuous functions they will agree at 0. So, $g^{\#}_x(0) = \sum_{n\in\mathbb{Z}} \hat{S}(x + n)$ or $S(0) = \sum_{n\in\mathbb{Z}} \hat{S}(x + n)$.

We will also need the following convolution result to prove our theorem.

**Theorem 3.3** *Suppose $G$ is a locally compact abelian group. Let $1 < r < 2$. Then $L^r * L^{r'}(\hat{G}) \subseteq M_p(\hat{G})$, where $\frac{2r}{3r-2} \leq p \leq \frac{2r}{2-r}$ and $\frac{1}{r} + \frac{1}{r'} = 1$.*

The above result is given in [31, page 126]. The main ingredient of the proof is the use of a Multilinear Riesz-Thorin Interpolation theorem. For completeness we give the details of the proof.

Let $\mathcal{S}(G)$ denote the set of Haar integrable simple functions on $G$. If $\phi,\ \psi \in \mathcal{S}(G)$, we will estimate the norm of the translation invariant operator $T_{\phi*\psi}$ acting on $L^p(G)$ by $(T_{\phi*\psi}f)^\wedge(\gamma) = \phi * \psi(\gamma)\hat{f}(\gamma) \quad \forall f \in \mathcal{S}(G)$. If $g \in \mathcal{S}(G)$, then $\langle T_{\phi*\psi}f, g\rangle \equiv U(\phi,\psi,f,g) = \int_{\hat{G}} \phi*\psi(\gamma)\hat{f}(\gamma)\hat{g}(\gamma)d\gamma$. First note that Hölder's inequality and Plancherel theorem will imply

$$\begin{aligned} |U(\phi,\psi,f,g)| &\leq \|\phi*\psi\|_{L^\infty(\hat{G})}\|\hat{f}\|_{L^2(\hat{G})}\|\hat{g}\|_{L^2(\hat{G})} \\ &\leq \|\phi\|_{L^1(\hat{G})}\|\psi\|_{L^\infty(\hat{G})}\|f\|_{L^2(G)}\|g\|_{L^2(G)}. \end{aligned}$$

Also by Parseval formula and Plancherel theorem we have

$$\begin{aligned} |U(\phi,\psi,f,g)| &= \left|\int_G \check{\phi}(x)\check{\psi}(x) f*g(x)dx\right| \\ &\leq \|\phi\|_{L^2(\hat{G})}\|\psi\|_{L^2(\hat{G})}\|f\|_{(L^1(G)}\|g\|_{L^\infty(G)}. \end{aligned}$$

These two inequalities say that $U$ extends to a bounded multilinear operator from

$$L^1(\hat{G}) \times L^\infty(\hat{G}) \times L^2(G) \times L^2(G) \to \mathbb{C}$$

and from

$$L^2(\hat{G}) \times L^2(\hat{G}) \times L^1(G) \times L^\infty(G) \to \mathbb{C}.$$

Hence from the Multilinear Riesz-Thorin Interpolation theorem we get

$$|U(\phi,\psi,f,g)| \leq \|\phi\|_r\|\psi\|_{r'}\|f\|_p\|g\|_{p'}$$

where $\frac{1}{r} = (1-t).1 + \frac{t}{2}$, $\frac{1}{p} = \frac{1-t}{2} + t.1$, and $0 \leq t \leq 1$. Hence, $L^r(\hat{G}) * L^{r'}(\hat{G}) \subset M_p(\hat{G})$, for $p = \frac{2r}{3r-2}$ and $1 < r < 2$. Furthermore, it is well known that, for $1 < p < 2$, $M_p(\hat{G}) \simeq M_{p'}(\hat{G})$. So, if $m \in M_p(\hat{G})$ and $T_m$ is the multiplier operator corresponding to $m$ then by Riesz-Thorin convexity theorem, $T_m : L^s(G) \to L^s(G)$, is a bounded linear operator with $\|T_m\|_s \leq \|m\|^{1-\alpha}_{M_p(\hat{G})}\|m\|^{\alpha}_{M_{p'}(\hat{G})}$ whenever $\frac{1}{s} = \frac{1-\alpha}{p} + \frac{\alpha}{p'}$, where $0 \leq \alpha \leq 1$. Notice that if $p < s < p'$ then there exist $\alpha \in (0,1)$ such that $\frac{1}{s} = \frac{1-\alpha}{p} + \frac{\alpha}{p'}$. Thus, we have $\phi * \psi \in M_q(\hat{G})$, where $\frac{2r}{3r-2} < q < \frac{2r}{2-r}$.

**Remark:** For $r = 2$, $L^2 * L^2(G) \subseteq M_p(\hat{G})$, $\forall p \in [1, \infty)$.

Finally we will also need the following theorem due to Berkson and Gillespie (see Theorem 1.10).

**Theorem 3.4** [14] *Let $\phi \in L^\infty(\hat{\mathbb{R}})$ and suppose that supp $T_\phi \subseteq [\frac{1}{4}, \frac{3}{4}]$. Then there is a unique bounded continuous function $\phi_c$ on $\mathbb{R}$ such that $\phi_c = \phi$ a.e. on $\mathbb{R}$. Suppose $1 \leq p < \infty$, then $\phi \in M_p(\hat{\mathbb{R}})$ if and only if $\phi_c|_{\mathbb{Z}} \in M_p(\mathbb{Z})$. If this is the case then*

$$\|\phi_c|_{\mathbb{Z}}\|_{M_p(\mathbb{Z})} \leq \|\phi\|_{M_p(\hat{\mathbb{R}})} \leq 2^{\frac{1}{p^*}} \|\phi_c|_{\mathbb{Z}}\|_{M_p(\mathbb{Z})},$$

*where $p^* = \max(p, p')$.*

**Proof of Theorem 3.2:** Let $0 < r < 1$ and assume $1 < p < 2$. Define $k_r(x) = \sum_{n \in \mathbb{Z}} \phi(n) r^{|n|} e^{2\pi i n x}$ for $x \in [0, 1)$. Then $k_r \in L^1(\mathbb{T})$ and $\hat{k}_r(n) = \phi(n) r^{|n|}$. Thus $\hat{k}_r \in l_1(\mathbb{Z})$ and

$$\|\hat{k}_r\|_{p'} \leq \|\phi\|_{p'}.$$

Define $F_r(x) = k_r(x) S(x)$ for $x \in [0, 1)$. Clearly $F_r \in L^1(\mathbb{R})$ and *supp* $F_r \subseteq [\frac{1}{4}, \frac{3}{4}]$ and $\hat{F}_r(\xi) = \sum_n \phi(n) \hat{S}(\xi - n) r^{|n|}$. Then $\hat{F}_r|_{\mathbb{Z}}(l) = \hat{k}_r * (S^\#)^\wedge(l)$. So by Theorem 3.3, $\hat{F}_r|_{\mathbb{Z}} \in M_q(\mathbb{Z})$ for $q \in [\frac{2p}{3p-2}, \frac{2p}{2-p}]$ with $\|\hat{F}_r|_{\mathbb{Z}}\|_{M_q(\mathbb{Z})} \leq C_p \|\phi\|_{p'} \tau_p$. Hence, by Theorem 3.4, $\hat{F}_r \in M_q(\hat{\mathbb{R}})$ for $q \in [\frac{2p}{3p-2}, \frac{2p}{2-p}]$ with

$$\|\hat{F}_r\|_{M_q(\hat{\mathbb{R}})} \leq C_p \tau_p \|\phi\|_{p'}. \tag{3.2}$$

Again from Lemma 3.1 and dominated convergence theorem we have $\hat{F}_r(\xi) \to W_{\phi,\hat{S}}(\xi)$ a.e. as $r \to 1$. Therefore from Eqn. (3.2), we have $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$ and $\|W_{\phi,\hat{S}}\|_{M_q(\hat{\mathbb{R}})} \leq C_p \tau_p \|\phi\|_{p'}$ for $q \in [\frac{2p}{3p-2}, \frac{2p}{2-p}]$. Similarly for $2 < p < \infty$, by the same argument we get

$$W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$$

for $q \in [\frac{2p}{p+2}, \frac{2p}{p-2}]$ and

$$\|W_{\phi,\hat{S}}\|_{M_q(\hat{\mathbb{R}})} \leq C_p \tau_p \|\phi\|_{p'}.$$

This completes the proof of the theorem.

We will now relax the hypothesis that *supp* $S \subseteq [\frac{1}{4}, \frac{3}{4}]$ to allow $S$ to have arbitrary compact support by imposing a certain extra condition on $S$. Suppose *supp* $S \subseteq [-N, N]$ and $\sum_{n\in\mathbb{Z}} |\hat{S}(\xi + n)|^p < \infty$ for all $\xi \in [0, 1)$. Define $S_N(x) = S(4Nx - 2n)$. Then *supp* $S_N \subseteq [\frac{1}{4}, \frac{3}{4}]$. Also from the condition on $\hat{S}$ we have $\sum_{n\in\mathbb{Z}} |(S_N^{\#})^{\wedge}|^p < \infty$. Thus if $\phi \in l_{p'}(\mathbb{Z})$ from Theorem 3.2 we have $W_{\phi,\hat{S}_N} \in M_q(\hat{\mathbb{R}})$ for the values of $q$ mentioned in the statement of the theorem. This along with Lemma 4.6 of Chapter 4 says that $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$. So in particular

**Corollary 3.1** *Let* $S \in C_C^1(\mathbb{R})$ *and* $1 < p < 2$. *Then for* $\phi \in l_{p'}$, $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$ *for* $q \in [\frac{2p}{3p-2}, \frac{2p}{2-p}]$.

By putting additional restrictions on $\phi$ we have the following (note that $l_p(\mathbb{Z}) \subset l_{p'}(\mathbb{Z})$):

**Proposition 3.1** *Let* $1 < p < 2$. *Suppose* $S \in L^p(\mathbb{R})$ *and has compact support. For* $\phi \in l_p(\mathbb{Z})$, *define* $W_{\phi,\hat{S}}(\xi) = \sum_n \phi(n)\hat{S}(\xi - n)$. *Then* $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$ *for* $1 \leq q < \infty$ *and* $||W_{\phi,\hat{S}}||_{M_q(\hat{\mathbb{R}})} \leq C||\phi||_p||S||_p$.

**Proof:** Let *supp* $S \subseteq [-N, N]$ for some $N \in \mathbb{N}$. Define $S_N(x) = S(4Nx - 2N)$. Then *supp* $S_N \subseteq [\frac{1}{4}, \frac{3}{4}]$ and $S_N \in L^p(\mathbb{R})$. Let $S_N^{\#}$ be 1-periodic extension of $S_N$ from $[0, 1)$ and $(S_N^{\#})^{\wedge} \in l_{p'}(\mathbb{Z})$. Now, if $\phi \in l_p(\mathbb{Z})$ then $\phi * (S_N^{\#})^{\wedge} \in M_q(\mathbb{Z})$ for $1 \leq q < \infty$, because of the following reason. Consider the operator

$$T : l_1(\mathbb{Z}) \times L^1(\mathbb{T}) \longrightarrow M_q(\mathbb{Z})$$

$$T : l_2(\mathbb{Z}) \times L^2(\mathbb{T}) \longrightarrow M_q(\mathbb{Z}),$$

defined by $T(\phi, f) = \phi * \hat{f}$. Then $||T(\phi, f)||_{M_q(\mathbb{Z})} \leq ||\phi||_{l_p(\mathbb{Z})}||f||_{L^p(\mathbb{T})}$ for $p = 1$ or 2. So by Multilinear Riesz-Thorin interpolation theorem, $T$ is a bounded and multilinear operator from $l_p(\mathbb{Z}) \times L^p(\mathbb{T})$ into $M_q(\mathbb{Z})$ for $1 < p < 2$ and $q \in [1, \infty)$. Thus $\phi * (S_N^{\#})^{\wedge} \in M_q(\mathbb{Z})$. Following the same approach as in the proof of Theorem 3.2 we have $W_{\phi,\hat{S}_N} \in M_q(\hat{\mathbb{R}})$. So by

Lemma 4.6 of Chapter 4 we have $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$ and $\|W_{\phi,\hat{S}}\|_{M_q(\mathbb{R})} \leq C\|\phi\|_p\|S\|_p$, where $C$ is a constant depending on support of $S$.

**Remark:** Observe that our result (Theorem 3.2) does not match with Jodeit's result (Theorem 3.1) in the limiting case $p = 1$. In our case $\phi$ is just a bounded sequence but Jodeit considered $\phi$ to be in $M_q(\mathbb{Z}) = l_\infty(\mathbb{Z}) \cap M_q(\mathbb{Z})$. For the case $p = 2$, we have $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$ for all $q \in [1,\infty)$ whenever $\phi \in l_2(\mathbb{Z})$. From Plancherel theorem it is easy to see that $l_2(\mathbb{Z}) = l_2(\mathbb{Z}) \cap M_q(\mathbb{Z})$ for all $q \in [1,\infty)$. These observations pose the following problem:

" Let $S \in L^1(\mathbb{R})$, *supp* $S \subseteq [\frac{1}{4},\frac{3}{4}]$ and $\sum\limits_{n\in\mathbb{Z}}|(S^\#)^\wedge(n)|^p < \infty$, for $1 < p < \infty$. For $\phi \in l_p(\mathbb{Z}) \cap M_q(\mathbb{Z})$, define $W_{\phi,\hat{S}} = \sum\limits_{n\in\mathbb{Z}}\phi(n)\hat{S}(\xi - n)$. Then is it true that $W_{\phi,\hat{S}} \in M_q(\hat{\mathbb{R}})$?"

## 3.3 A Maximal Inequality

In this section we will prove a maximal inequality for a sequence of functions $S_j$ as in Theorem 3.2. For this we need the following Lemma. In [1] Asmar, Berkson and Gillespie proved it for sequences.

**Lemma 3.2** *Let $G$ be a locally compact abelian group. Suppose $1 < p < \infty$ , $N \in \mathbb{N}$ and $\{\phi_j\}_{j=1}^N \subseteq M_p(\hat{G})$. Define $Mf(x) = \sup\limits_{1\leq j\leq N} |T_{\phi_j}f(x)|$. Let $\{\phi_{j,r}\}_{r\in I}$ be a net in $M_p(\hat{G})$, for the index set I=[0,1], such that*

*(i) for $1 \leq j \leq N$, $\phi_{j,r} \to \phi_j$ as $r \to 1$ pointwise on $\hat{G}$,*

*(ii)* $\sup\limits_{1\leq j\leq N} \{|\phi_{j,r}(\gamma)| : 1 \leq j \leq N, r \in I, \gamma \in \hat{G}\} < \infty$,

*(iii)* $\sup\limits_{r\in I} \|M^r\|_p = C < \infty$,

*where $C$ is a constant and $M^r$ is the maximal operator ( with $(p,p)$ norm $\|M^r\|_p$) corresponding to the multiplier transform $\{T_{\phi_{j,r}}\}_{j=1}^N$. Then $\|M\|_p \leq C$.*

Let $\{S_j\}_{j\in\mathbb{N}} \subseteq L^1(\mathbb{R})$ with *supp* $S_j \subseteq [\frac{1}{4}, \frac{3}{4}]$ and $\sum_n |\hat{S}_j(n)|^p < \infty$. The maximal operator which is associated with this sequence is defined as

$$Mf(x) = \sup_j |T_{W_{\phi,\hat{S}_j}} f(x)|, \quad \text{for } \phi \in l_{p'}(\mathbb{Z}) \quad \text{and } f \in L^p(\mathbb{R}).$$

We know that $W_{\phi,\hat{S}_j} \in M_q(\hat{\mathbb{R}})$ for some values of $q$. Also $W_{\phi,\hat{S}_j|_\mathbb{Z}} \in M_q(\mathbb{Z})$. Let us define the maximal operator associated with $W_{\phi,\hat{S}_j|_\mathbb{Z}}$ as

$$\tilde{M}f(x) = \sup_j |T_{W_{\phi,\hat{S}_j|_\mathbb{Z}}} f(x)| \quad \textit{for } f \in L^p(\mathbb{T}).$$

We have the following result.

**Theorem 3.5** *Suppose $\{S_j\}_{j\in\mathbb{Z}}$ satisfies the above mentioned conditions. Then $\|M\|_p \leq C$ if $\|\tilde{M}\|_p \leq C$, where $C$ is a constant.*

**Proof:** Let us assume $\phi \in l_{p'}(\mathbb{Z})$ is finitely supported. We have $W_{\phi,\hat{S}_j|_\mathbb{Z}} = \phi * \hat{S}_j^{\#}$, where $S_j^{\#}$ is the 1-periodic extension of $S_j$ from $[0,1)$. Define $k(x) = \sum_{n\in\mathbb{Z}} \phi(n)e^{2\pi inx}$ for $x \in [0,1)$ and $F_j(x) = k^{\#}(x)S_j(x)$, where $k^{\#}$ is the 1-periodic extension of $k$. Then $F_j \in L^1(\mathbb{R})$ and *supp* $F_j \subseteq [\frac{1}{4}, \frac{3}{4}]$, $\forall j \in \mathbb{N}$. Moreover,

$$\begin{aligned}
\hat{F}_j(\xi) &= \int_{\frac{1}{4}}^{\frac{3}{4}} k(x)S_j(x)e^{-2\pi ix\xi}dx \\
&= \sum_{n\in\mathbb{Z}} \phi(n) \int_{\frac{1}{4}}^{\frac{3}{4}} S_j(x)e^{-2\pi ix(\xi-n)}dx \\
&= W_{\phi,\hat{S}_j}.
\end{aligned}$$

Therefore,

$$T_{W_{\phi,\hat{S}_j}} f(x) = F_j * f \quad \textit{for } f \in L^q(\mathbb{R}).$$

where $q$ belongs to the same range as given in Theorem 3.2. For $f \in L^q(\mathbb{R})$ and for each $n \in \mathbb{N}$ consider $f_n$, a periodic function such that $f_n(x) = f(x+n)$ for $x \in [0,1)$. For

$x \in [\frac{1}{4}, \frac{3}{4}]$

$$
\begin{aligned}
T_{W_{\phi,\hat{S}_j}} f(x+n) &= \int_{\frac{1}{4}}^{\frac{3}{4}} k^{\#}(y) S_j(y) f(x-y+n)dy \\
&= \int_{\frac{1}{4}}^{\frac{3}{4}} k(y) S_j^{\#}(y) f_n(x-y)dy \\
&= T_{W_{\phi,\hat{S}_j|_{\mathbb{Z}}}} f_n(x).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_n \int_0^1 (\sup_j |T_{W_{\phi,\hat{S}_j}} f(x+n)|)^q dx &= \sum_n \int_0^1 (\sup_j |T_{W_{\phi,\hat{S}_j|_{\mathbb{Z}}}} f_n(x)|)^q dx \\
&= \sum_n \|\tilde{M} f_n\|^q_{L^q(\mathbb{T})} \le C^q \sum_n \|f_n\|^q_{L^q(\mathbb{T})}.
\end{aligned}
$$

So,

$$
\|Mf\|_q^q \le C^q \sum_n \|f_n\|_q^q = C^q \|f\|_q^q.
$$

For arbitrary $\phi \in l_{p'}$, let $\phi_l = \chi_{[-l,l]\cap\mathbb{Z}}\phi$. Then $W_{\phi_l,\hat{S}_j} \to W_{\phi,\hat{S}_j}$ pointwise as $l \to \infty$ and $|W_{\phi_l,\hat{S}_j}(\xi)| \le C\|\phi\|_{p'}\tau_p$ and $\|M^l f\|_q \le C\tau_p\|\phi\|_{p'}$, where $M^l$ is the maximal operator associated with $T_{W_{\phi_l,\hat{S}_j}}$. So by Lemma 3.1, $\|Mf\|_q \le C\|\phi\tau_p\|_{p'}\|f\|_q$.

# Chapter 4

# Weak-Type Multipliers

## 4.1 Introduction

In this chapter we will study weak-type $(p,p)$ multipliers. Questions regarding "extension problems" can also be asked in this particular setting. The dual problem dealing with "restrictions" of weak-type $(p,p)$ multipliers has been widely studied. To put things in prespective, we feel the need to begin with a brief survey of "restriction problems" in §2 of this chapter. In §3, we will prove some results concerning extensions of weak-type multipliers.

Let $T$ be a translation invariant operator on $L^p(G)$, where $G$ is a locally compact abelian group, with Haar measure $\lambda$. There are many translation invariant operators such as the Hilbert transform, Calderón-Zygmund singular integral operators on $\mathbb{R}^N$, which are not bounded on $L^1$ but satisfy weak $(1,1)$ inequalities. So it is natural to study the translation invariant operators satisfying weak $(p,p)$ inequalities i.e., $T : L^p(G) \longrightarrow L^p(G)$ such that $\tau_x T = T\tau_x \ \forall x \in G$, and for which there exists a constant $C$ such that

$$\lambda\{x \in G : |Tf(x)| > t\} \leq \frac{C^p}{t^p}||f||_p^p \tag{4.1}$$

for all $f \in L^p(G)$ and $t > 0$. We denote by $M(L^p, L^{p,\infty})$ the set of all such operators. Of course, if $T \in M(L^p(G))$ then $T \in M(L^p, L^{p,\infty})$. It is well known that if $T \in M(L^p(G))$ then

there exists a function $\phi \in L^\infty(\hat{G})$ such that $T = T_\phi$ where $(T_\phi f)^\wedge = \phi\hat{f}\ \ \forall f \in L^2 \cap L^p(G)$. We say that $\phi$ is a multiplier of weak-type $(p,p)$ if

$$\lambda\{x \in G : |T_\phi f(x)| > t\} \leq \frac{C^p}{t^p}\|f\|_p^p \tag{4.2}$$

for all $f \in L^2 \cap L^p(G)$ and $t > 0$. So by Chebyshev inequality if $\phi \in M_p(\hat{G})$ then $\phi$ is a multiplier of weak-type $(p,p)$. Then, if the operator $T_\phi$ satisfies Eqn. (4.2), it extends uniquely from $L^2 \cap L^p(G)$ to a linear mapping $T_\phi^{(p)}$ on $L^p(G)$ into the space of $\lambda$-measurable functions on $G$ such that whenever $\{f_n\}_{n=1}^\infty$ converges to $f$ in $L^p(G)$, then $\{T_\phi f_n\}_{n=1}^\infty$ converges in measure to $T_\phi^{(p)} f$. Clearly $T_\phi^{(p)}$ is translation invariant and the inequality in Eqn. (4.2) remains valid (with the same constant $C$) for all $f \in L^p(G)$ and $t > 0$.

For $1 \leq p < \infty$ denote $M_p^{(w)}(\hat{G}) = \{\phi :\ \lambda\{x \in G : |T_\phi f(x)| > t\} \leq \frac{C^p}{t^p}\|f\|_p^p\}$ ( the constant $C$ may depend on $\phi$). For $\phi \in M_p^{(w)}(\hat{G})$, let $N_p^{(w)}(\phi)$ be the smallest constant $C \geq 0$ such that Eqn. (4.2) holds. $N_p^{(w)}(\phi)$ is the weak-type "norm" of $T_\phi^{(p)}$ on $L^p(G)$ ( which is not a norm but a quasi norm. In fact $N_p^{(w)}(\phi + \psi) \leq 2(N_p^{(w)}(\phi) + N_p^{(w)}(\psi))$ for $\phi, \psi \in M_p^{(w)}(\hat{G})$).

The study of multipliers of weak-type $(p,p)$ was initiated by Misha Zafran in [41]. As mentioned above every multiplier operator $T_\phi$ on $L^p(G)$ satisfies a weak-type $(p,p)$ inequality. So the question arises "Does there exits a $\phi \in M_p^{(w)}(\hat{G})$, for $1 < p < \infty$ such that $\phi \notin M_p(\hat{G})$?" In 1975 Zafran [41] produced examples of functions $\phi$ such that $\phi \in M_p^{(w)}(\hat{G}) \cap C_0(\hat{G})$, for $1 < p < 2$ when $G = \mathbb{R}^N, \mathbb{T}^N, \mathbb{Z}^N$ or $\mathbb{R}_b^N$ where $\mathbb{R}_b^N$ is the Bohr compactification of $\mathbb{R}^N$, but $\phi \notin M_p(\hat{G})$.

We mentioned earlier that $N_p^{(w)}(.)$ is not a norm but a quasi norm. So, we expect $M_p^{(w)}(\hat{G})$ to be a quasi Banach space. To prove this we need to find the relation between $\|.\|_\infty$ and $N_p^{(w)}(.)$. In [8], Asmar, Berkson, and Gillespie proved the following result. A simple application of this result yeilds that $M_p^{(w)}(\hat{G})$ is a quasi Banach space.

**Theorem 4.1** *If $1 \leq p < \infty$ and $\phi \in M_p^{(w)}(\hat{G})$ then*

$$\|\phi\|_\infty \leq K_p N_p^{(w)}(\phi),$$

*where $K_p$ is a real constant which depends only on $p$.*

They proved this theorem by using the homomorphism theorem for weak multipliers and the fact that if $k \in L^1(\hat{G})$ and $\phi \in M_p^{(w)}(\hat{G})$ then $k * \phi \in M_p^{(w)}(\hat{G})$ [7], for $1 < p < \infty$. See Rapaso [35] for the case $p = 1$ (we will discuss these results in the next section).

Now, to establish appropriate parallels between weak-type and strong type multipliers one has to answer the following question: "If $T$ is a translation invariant linear mapping of weak-type $(p,p)$ on $L^p(G)$, then does there exists a measurable function $\phi$ on $\hat{G}$ such that $T = T_\phi^{(p)}$. Asmar, Berkson and Gillespie [8] answered this question in the affirmative for $1 < p < \infty$. For $p \neq 2$, they showed that if $T \in M(L^p, L^{p,\infty})$, then

$$T(f * g) = Tf * g = f * Tg \tag{4.3}$$

for every $g \in L^p(G)$ and for every simple integrable function $f$. Further,

$$\|Tf\|_{p,\infty}^* \leq N_p^{(w)}(T)\|f\|_{p,1}^* \tag{4.4}$$

and

$$\|Tf\|_{p',\infty}^* \leq p' N_p^{(w)}(T)\|f\|_{p',1}^*. \tag{4.5}$$

By Marcinkiewicz interpolation theorem it follows that, $T$ is of strong type $(2,2)$ on the linear space of integrable simple functions. So $T = T_\phi^{(p)}$ for some $\phi \in M_p^{(w)}(\hat{G})$ on $L^p(G)$. For $p = 2$ they showed that, if $T$ is a translation invariant operator satisfying weak-type $(2,2)$ then $T$ is of strong type $(2,2)$ on $L^2(G)$.

## 4.2 The Restriction Problem For Multipliers Of Weak Type $(p, p)$

The analogue of de Leeuw's restriction problems, for multipliers of weak-type $(p, p)$, has been studied by various mathematicians. In this section we will make a brief survey of restriction problems for multipliers of weak-type $(p, p)$.

A generalization of de Leeuw's restriction theorem for arbitrary locally compact abelian groups is the homomorphism theorem for multipliers [23]. More precisely,

**Theorem 4.2** *Suppose $G_1$ and $G_2$ are two locally compact abelian groups. Let $\phi \in M_p(\hat{G}_1) \cap C(\hat{G}_1)$ and let $\rho : \hat{G}_2 \to \hat{G}_1$ be a continuous homomorphism. Then $\phi \circ \rho \in M_p(\hat{G}_2)$ and $\|\phi \circ \rho\|_{M_p(\hat{G}_2)} \leq C\|\phi\|_{M_p(\hat{G_1})}$, where $C$ is a constant which depends only on $p$.*

Hence if $\phi \in M_p(\hat{\mathbb{R}^N}) \cap C(\hat{\mathbb{R}}^N)$ and $\rho$ is the usual inclusion map from $\mathbb{Z}^N$ to $\mathbb{R}^N$, the homomorphism theorem will imply de Leeuw's theorem on $\mathbb{R}^N$. Analogous results for multipliers of weak-type $(p, p)$ for $1 \leq p < \infty$ have been proved by Asmar, Berkson and Gillespie [7] [6]. In order to prove this, they proved results regarding convolution of an $L^1$ function with a multiplier of weak-type $(p, p)$. These results are interesting in themselves and we will discuss these here.

Let $(X, \nu)$ be an arbitrary measure space and $u \mapsto R_u$ a strongly continuous representation of $G$ on $L^p(X)$ $(1 \leq p < \infty)$ satisfying the following conditions:

(a) $R$ is separation preserving on $L^p(X)$ (i.e. if $f, g \in L^p(X)$ and $fg = 0$ a.e., then for all $u \in G$, $(R_u f)(R_u g) = 0$ a.e.).

(b) There is a positive real constant $K_p$ such that $\|R_u f\|_p \leq K_p \|f\|_p$ for all $u \in G$ and $f \in L^p(X)$.

(c) There is a positive real constant $K_\infty$ such that $\|R_u f\|_\infty \leq K_\infty \|f\|_\infty$ for all $u \in G$ and $f \in L^p(X) \cap L^\infty(X)$.

Under these conditions it is proved in [5] that

$$\nu\{x \in X : |R_u f(x)| > t\} \leq (K_p K_\infty)^p \nu\{x \in X : |f(x)| > \frac{t}{K_\infty}\} \tag{4.6}$$

for all $u \in G$, $f \in L^p(X)$ and $t > 0$. Such representations are called $\nu$-distributionally controlled. In [3], Asmar, Berkson, and Gillespie showed that for each such $R$ there is a uniformly bounded strongly continuous repreesentation $R^{(2)}$ of $G$ acting on $L^2(X)$ such that for all $u \in G$ and $f \in L^2(X) \cap L^p(X)$, $R_u^{(2)} f = R_u f$. Now $\{R_u^{(2)}\}_{u \in G}$ is a bounded multiplicative group of operators on $L^2(X)$. So there exists a self adjoint invertible operator $U$ on $L^2(X)$ such that for every $u \in G$, $U R_u^{(2)} U^{-1}$ is unitary [20, Theorem 8.1]. Then by the generalized Stone's theorem [36], there exists a unique regular Borel spectral measure $\mathcal{E}(.)$ on $\hat{G}$ such that

$$R_u^{(2)} = \int_{\hat{G}} \gamma(u) d\mathcal{E}(\gamma) \quad \forall u \in G. \tag{4.7}$$

Let $\Theta : \hat{G} \longrightarrow \mathbb{C}$ be a bounded Borel measurable function. Define a bounded linear operator $\Im_\Theta : L^2(X) \longrightarrow L^2(X)$ by

$$\Im_\Theta = \int_{\hat{G}} \Theta(\gamma) d\mathcal{E}(\gamma).$$

If $\Im_\Theta$ satisfies weak-type $(p,p)$ inequality for $f \in L^2 \cap L^p(X)$ then it has a unique extension from $L^2 \cap L^p(X)$ to a linear mapping $\Im_\Theta^{(p)}$ from $L^p(X)$ to the space of complex valued measurable functions on $X$. If $\Theta = \hat{k}$ for some $k \in L^1(G)$ then

$$\Im_{\hat{k}}^{(p)} f = H_k f \;\; \forall f \in L^p(X), \tag{4.8}$$

where $H_k$ is the transferred operator defined by

$$H_k f(.) = \int_G k(u) R_{u^{-1}} f(.) du \;\; \forall f \in L^p(X).$$

By considering $X = G$ and the representation $R$ given by $R_u f(x) = f(ux)$ for $u, x \in G$ and $f \in L^p(G)$, it is easy to see that if $\phi \in M_p(\hat{G})$ then $\Im_\phi = T_\phi$, where $T_\phi$ is the multiplier operator corresponding to $\phi$.

In [4], Asmar, Berkson, and Gillespie proved the following theorem for $1 < p < \infty$ and in [6] they proved it for the case $p = 1$.

**Theorem 4.3** [4, Theorem 2.6] [6, Theorem 6.5] *Suppose that* $1 \leq p < \infty$ , *and* $\{\phi_j\}_{j\geq 1} \subseteq M_p^{(w)}(\hat{G}) \cap C(\hat{G})$. *Then for each* $j$, *the operator* $\Im_{\phi_j}^{(p)}$ *can be defined as above on* $L^p(X)$ *and*

$$\nu\{x \in X : \sup_{j\geq 1} |\Im_{\phi_j}^{(p)} f(x)| > t\} \leq (\frac{1}{t}(K_p K_\infty)^2 (p')^2 N_p^{(w)}(\{\phi_j\}_{j\geq 1})\|f\|_p)^p$$

*for all* $f \in L^p(X)$ *and all* $t > 0$, *where* $p'$ *is the index conjugate to* $p$ , *and* $K_p$ *and* $K_\infty$ *are constants occuring in (b) and (c).*

They proved the above theorem by applying the following convolution result.

**Lemma 4.1** [4] [6] *Suppose that* $1 \leq p < \infty$, $k \in L^1(\hat{G})$, *and* $\{\phi_j\}_{j\geq 1}$ *is a (finite or infinite) sequence of functions belonging to* $M_p^{(w)}(\hat{G})$. *Then* $\{k * \phi_j\}_{j\geq 1} \subseteq M_p^{(w)}(\hat{G})$, *and*

$$N_p^{(w)}(\{k * \phi_j\}_{j\geq 1}) \leq p'\|k\|_1 N_p^{(w)}(\{\phi_j\}_{j\geq 1}),$$

*where* $N_p^{(w)}(\{\phi_j\}_{j\geq 1}$ *denotes the weak-type* $(p,p)$ *norm of the maximal operator defined on* $L^p(G)$ *by the sequence* $\{T_{\phi_j}^{(p)}\}_{j\geq 1}$.

From Theorem 4.3 we can deduce the homomorphism theorem for $1 \leq p < \infty$. Let $\rho : \hat{G}_1 \longrightarrow \hat{G}_2$ be a continuous homomorphism and $\hat{\rho} : G_2 \longrightarrow G_1$, the dual homomorphism defined by $(\hat{\rho}(u))(\gamma) = u(\rho(\gamma))$ for all $u \in G_2$ and $\gamma \in \hat{G}_1$. For $1 \leq p < \infty$, define $R_u^{(p)}$ on $L^p(G_1)$ by $R_u^{(p)} f(x) = f(\hat{\rho}(u)x)$. So $R_u^{(p)}$ is a strongly continuous distributionally controlled representation of $G_2$ on $L^p(G_1)$, with $K_p = K_\infty = 1$. Also for each bounded Borel measurable function $\Theta$ on $G_2$, the operator $\Im_\Theta$ is the $L^2(G_1)$ multiplier transform corresponding to the bounded measurable function $\Theta \circ \rho$ on $\hat{G}_1$ [2]. So, if $\phi \in M_p^{(w)}(\hat{G}_2)$ then $\Im_\phi f = T_{\phi\circ\rho} f$ for $f \in L^2 \cap L^p(G_1)$. Hence from Theorem 4.3, we have $\phi \circ \rho \in M_p^{(w)}(\hat{G}_1)$. So we have the homomorphism theorem for weak-type multipliers. This is formally stated below:

**Theorem 4.4** [4, Theorem 4.1] [6, Theorem 1.5] *Suppose* $1 \leq p < \infty$, *and* $\rho$ *is a continuous homomorphism from* $\hat{G}_1$ *to* $\hat{G}_2$. *Let* $\{\phi_j\}_{j\geq 1} \subseteq M_p^{(w)}(\hat{G}_2) \cap C(\hat{G}_2)$. *Then* $\phi_j \circ \rho \in M_p^{(w)}(\hat{G}_1)$ *for each* $j \geq 1$ *and* $N_p^{(w)}(\{\phi_j \circ \rho\}_{j\geq 1}) \leq (p')^2 N_p^{(w)}(\{\phi\}_{j\geq 1})$.

In [9], Asmar,Berkson, and Bourgain answered the following question posed by A.Pelczyǹski [34] " If $\phi \in M_1^{(w)}(\hat{\mathbb{R}^N})$ and is continuous at each point of $\mathbb{Z}^N$, is it necessarily true that $\phi|_{\mathbb{Z}^N}$ belongs to $M_1^{(w)}(\hat{\mathbb{Z}}^N)$?" The answer follows from the following theorem which they proved in the same paper.

**Theorem 4.5** [9, Theorem 1.2] *Suppose* $k \in L^1(\hat{\mathbb{R}^N})$ *and* $\{\phi_j\}_{j\geq 1} \subseteq M_1^{(w)}(\hat{\mathbb{R}^N})$ *then* $k * \phi_j \in M_1^{(w)}(\hat{\mathbb{R}^N})$ *for each* $j \geq 1$ *and* $N_1^{(w)}(\{k * \phi_j\}_{j\geq 1}) \leq K_N \|k\|_1 N_1^{(w)}(\{\phi_j\}_{j\geq 1})$ *where* $K_N$ *is a constant depending only on* $N$.

In [35], Raposo extended Theorem 4.5, the result of Asmar, Berkson and Bourgain [9] to a more general set up. He proved the following:

**Theorem 4.6** *Let* $\lambda \in M(G)$ *and* $\{\phi_j\}_{j\geq 1} \subseteq M_1^{(w)}(\hat{G})$. *Then* $\lambda * \phi_j \in M_1^{(w)}(\hat{G})$ *for each* $j \geq 1$ *and*

$$N_1^{(w)}(\{\lambda * \phi_j\}_{j\geq 1}) \leq C\|\lambda\| N_1^{(w)}(\{\phi_j\}_{j\geq 1}),$$

*where* $C$ *is a universal constant.*

If each $\{\phi_j\}_{j=1}^J$ for some $J \in \mathbb{N}$, has compact support define $h_{n,j} \in L^1(\mathbb{R}^N)$ such that $\hat{h}_{n,j} = k_n * \phi_j$, where $k_n$ is the n-th Fejér kernel on $\mathbb{R}^N$. Let $R_x f(t) = f(t\theta(x))$ for $f \in L^1(\mathbb{T}^N)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$ and $\theta(x) = (e^{2\pi i x_1}, ..., e^{2\pi i x_n})$. Then the transferred convolution operator $H_{h_{n,j}}$ on $L^1(\mathbb{T}^N)$ is defined by $H_{h_{n,j}} f = \int\limits_{\mathbb{R}^N} h_{n,j}(x) R_{-x} f dx \ \ \forall f \in L^1(\mathbb{T}^N)$, and also satisfies

$$H_{h_{n,j}} f = T_{\hat{h}_{n,j}|_{\mathbb{Z}^N}} f = T_{(k_n * \phi_j)|_{\mathbb{Z}^N}} f.$$

As $R$ is a distributionally bounded representation we have

$$|\{x \in \mathbb{T}^N : \max_{1\leq j\leq J} |T_{(k_n*\phi_j)|_{\mathbb{Z}^N}} f(x)| > t\}| \leq \frac{N_1^{(w)}(\{k_n * \phi_j\}_{j=1}^J)}{t} \|f\|_1.$$

Then by usual approximation one gets result for any $\phi \in M_1^{(w)}(\hat{\mathbb{R}^N})$. In [9], Asmar, Berkson, and Bourgain proved the maximal inequality for the following theorem.

**Theorem 4.7** *Suppose $\phi \in M_1^{(w)}(\hat{\mathbb{R}^N})$ and continuous at each point on $\mathbb{Z}^N$. Then $\phi|_{\mathbb{Z}^N} \in M_1^{(w)}(\mathbb{Z}^N)$ and $N_1^{(w)}(\phi|_{\mathbb{Z}^N}) \leq K_N N_1^{(w)}(\phi)$, where $K_N$ depends only on $N$.*

# 4.3 Weak-Type Extensions

## 4.3.1 Introduction

In this section we are concerned with extensions of weak-type multipliers from $\mathbb{Z}^N$ to $\mathbb{R}^N$ through summability kernels. Let us define summability kernels for weak-type multipliers as follows:

**Definition 4.1** *A bounded measurable function $\Lambda: \mathbb{R}^N \longrightarrow \mathbb{C}$ is called a weak summability kernel for $M_p^{(w)}(\hat{\mathbb{R}}^N)$ if for every $\phi \in M_p^{(w)}(\mathbb{Z}^N)$, the function $W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}^N} \phi(n)\Lambda(\xi - n)$ is defined a.e. and belongs to $M_p^{(w)}(\hat{\mathbb{R}}^N)$.*

This definition is just the weak-type analogue of summability kernels for strong type multipliers. We first cite two important results regarding the summability kernels of strong type multipliers from the work of Jodeit [29] and of Berkson, Paluszynski and Weiss [11]:

**Theorem 4.8** [29] *Let $S \in L^1(\mathbb{R}^N)$ and supp $S \subseteq [\frac{1}{4}, \frac{3}{4}]^N$ with $\tau = \sum_{n\in\mathbb{Z}^N} |(S^{\#})^{\wedge}(n)| < \infty$, where $S^{\#}$ is the 1-periodic extension of $S$ from $[0,1)$. Then the function defined by*

$$W_{\phi,\hat{S}}(\xi) = \sum_{n\in\mathbb{Z}^N} \phi(n)\hat{S}(\xi - n)$$

*belongs to $M_p(\hat{\mathbb{R}}^N)$, for $1 \leq p < \infty$ with*

$$\|W_{\phi,\hat{S}}\|_{M_p(\hat{\mathbb{R}}^N)} \leq C_p \tau \|\phi\|_{M_p(\mathbb{Z}^N)}.$$

**Theorem 4.9** [11] *For* $1 \leq p < \infty$, *let* $\Lambda \in M_p(\hat{\mathbb{R}}^N)$ *and supp* $\Lambda \subseteq [\frac{1}{4}, \frac{3}{4}]^N$. *For* $\phi \in M_p(\mathbb{Z}^N)$ *define*

$$W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}^N} \phi(n)\Lambda(\xi - n)$$

*on* $\mathbb{R}^N$. *Then* $W_{\phi,\Lambda} \in M_p(\hat{\mathbb{R}^N})$ *and*

$$||W_{\phi,\Lambda}||_{M_p(\hat{\mathbb{R}}^N)} \leq C_p||\Lambda||_{M_p(\hat{\mathbb{R}}^N)}||\phi||_{M_p(\mathbb{Z}^N)},$$

*where the constant* $C_p$ *depends on p and on the support of* $\Lambda$.

In fact, the condition *supp* $\Lambda \subseteq [\frac{1}{4}, \frac{3}{4}]^N$ in Theorem 4.9 can be relaxed to allow $\Lambda$ to have arbitrary compact support. Asmar, Berkson and Gillespie proved a weak-type analogue of Theorem 4.8 in [5]. In this same paper they also proved that, in the case when $\Lambda(\xi) = \prod_{j=1}^{N} \max(1 - |\xi_j|, 0)$ for $\xi = (\xi_1, ..., \xi_N)$, then $\Lambda$ is a weak-type summability kernel for $1 \leq p < \infty$. In fact this is a particular case of Theorem 4.9. In §4.3.3 we prove a more general result for $1 < p < \infty$. In §4.3.4 we relax the hypothesis that the support be compact. We will obtain weak-type inequalities by applying the technique of transference couples (as in [11]) to prove our main result. In §4.3.5, as an application of our result, we prove a weak-type analogue of an extension theorem by de Leeuw.

### 4.3.2 Weak-Type Inequality for Transference Couples

Let us recall the definition of transference couples.

**Definition 4.2** *For a locally compact group* $G$, *a transference couple is a pair* $(S, T) = (\{S_u\}, \{T_u\})$ , $u \in G$, *of strongly continuous mappings defined on* $G$ *with values in* $\mathcal{B}(X)$, *where* $X$ *is a Banach space, satisfying*

*(i)* $C_S = \sup\{||S_u|| : u \in G\} < \infty$

*(ii)* $C_T = \sup\{||T_u|| : u \in G\} < \infty$

*(iii)* $S_vT_u = T_{uv} \quad \forall u, v \in G.$

Let $\Lambda \in L^\infty(\hat{\mathbb{R}}^N)$ and *supp* $\Lambda \subseteq [\frac{1}{4}, \frac{3}{4}]^N$. Consider the following transference couple $(S, T)$ used by Berkson, Paluszyński and Weiss in [11]. For $u \in \mathbb{T}^N$ the family $T = \{T_u\}$ is given by

$$(T_uf)^\wedge(\xi) = \sum_{n\in\mathbb{Z}^N} \Lambda(\xi - n)e^{2\pi iu.n}\hat{f}(\xi) \ for \ f \in L^p(\mathbb{R}^N) \tag{4.9}$$

and the family $S = \{S_u\}$ is defined by

$$(S_uf)^\wedge(\xi) = \sum_{n\in\mathbb{Z}^N} b(\xi - n)e^{2\pi iu.n}\hat{f}(\xi) \ for \ f \in L^p(\mathbb{R}^N), \tag{4.10}$$

where $b(\xi) = \prod_{i=1}^{N} b_i(\xi_i)$ for $\xi = (\xi_1, \ldots\ldots, \xi_N)$ and for each $i$, $b_i$ is a continuous function defined on $\mathbb{R}$ as

$$b_i(x) = \begin{cases} 1 & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ 4x & \text{if } x \in [0, \frac{1}{4}) \\ 4(1-x) & \text{if } x \in (\frac{3}{4}, 1] \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$S_uf(x) = \sum_{l\in\mathbb{Z}^N} \check{\beta}_u(l)f(x + u - l) \quad a.e., \tag{4.11}$$

where $\check{\beta}_u$ is the Fourier transform of the function $\beta_u(\xi) = b(\xi)e^{2\pi i\xi.u}$ which is given explicitily by

$$\check{\beta}_u(\xi) = \prod_{i=1}^{N} \check{\beta}_{u_i}(\xi_i),$$

where

$$\check{\beta}_{u_i}(\xi_i) = \begin{cases} \frac{2e^{2\pi i(\frac{\xi_i+u_i}{2})}}{\pi^2(\xi_i-u_i)^2}(cos\frac{\pi}{2}(\xi_i - u_i) - cos\pi(\xi_i - u_i)) & \text{if } \xi_i \neq u_i \\ \frac{3e^{2\pi i(\frac{\xi_i+u_i}{2})}}{4} & \text{if } \xi_i = u_i. \end{cases}$$

For $l \in \mathbb{Z}^N$, an easy estimate shows that

$$|\check{\beta}_u(l)| \leq \beta(l) \quad \forall u \in \mathbb{T}^N \quad \text{where} \quad \beta(l) = \prod_{i=1}^{N} \beta_i(l_i) \text{ and}$$

$$\beta_i(l_i) = \begin{cases} \frac{1}{(l_i-1)^2} & if\ l_i > 1 \\ \frac{1}{(l_i+1)^2} & if\ l_i < 1 \\ \|b_i\|_1 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{l \in \mathbb{Z}^N} |\check{\beta}_u(l)| \leq \sum_{l \in \mathbb{Z}^N} \beta(l) = C < \infty. \tag{4.12}$$

In the following theorem we shall show that the operator transferred by $T$ (of the transfernce couple $(S,T)$ defined in Eqn. (4.9) and Eqn. (4.10)) given by

$$H_k f(.) = \int_{\mathbb{T}^N} k(u) T_{u^{-1}} f(.) du,$$

where $k \in L^1(\mathbb{T}^N)$ and $f \in L^p(\mathbb{R}^N)$, satisfies a weak $(p,p)$ inequality.

**Theorem 4.10** *Let $(S,T)$ be the transference couple as defined in Eqn. (4.9) and Eqn. (4.10). Then for $1 < p < \infty$ and $t > 0$*

$$\lambda\{x \in \mathbb{R}^N \ : \ |H_k f(x)| > t\} \leq \left(\frac{C\, C_p}{t} C_T N_p^{(w)}(k) \|f\|_p\right)^p,$$

*where $C = \sum_{l \in \mathbb{Z}^N} \beta(l)$ as in Eqn. (4.12), $C_T$ is the uniform bound for the family $T = \{T_u\}$, and $C_p = \frac{p}{p-1}$.*

**Proof:** Assume $f \in \mathcal{S}(\mathbb{R}^N)$. For $t > 0$ define $E_t = \{x \ : \ |H_k f(x)| > t\}$ and $\mathcal{F}_t = \{(v,x) \in \mathbb{T}^N \times \mathbb{R}^N \ : \ |\sum_{l \in \mathbb{Z}^N} \check{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x-l) du| > t\}$. Then

$$\begin{aligned} \lambda(E_t) &= \lambda\{x \in \mathbb{R}^N \ : \ |S_{v^{-1}} \textstyle\int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x) du| > t\} \\ &= \lambda\{x \in \mathbb{R}^N : |\textstyle\sum_{l \in \mathbb{Z}^N} \check{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x-v-l)| > t\} \\ &= \lambda\{x \in \mathbb{R}^N : |\textstyle\sum_{l \in \mathbb{Z}^N} \check{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x-l)| > t\} \end{aligned}$$

$$= \int_{\mathbb{T}^N} \lambda\{x \in \mathbb{R}^N : |\sum_{l\in\mathbb{Z}^N} \check{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x+l)| > t\}.$$

So from the definition of $\chi_{\mathcal{F}_t}$ and Fubini's theorem we have

$$\begin{aligned}
\lambda(E_t) &= \int_{\mathbb{T}^N}\int_{\mathbb{R}^N} \chi_{\mathcal{F}_t}(v,x)\,dx\,dv \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{T}^N} \chi_{\mathcal{F}_t}(v,x)\,dv\,dx \\
&= \int_{\mathbb{R}^N} |\{v : |\sum_{l\in\mathbb{Z}^N} \check{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x-l)| > t\}|\,dx,
\end{aligned}$$

where $|E|$ denotes the measure of the subset $E \subseteq \mathbb{T}^N$. Thus

$$\begin{aligned}
\lambda(E_t) &\leq \int_{\mathbb{R}^N} |\{v : \sum_{l\in\mathbb{Z}^N} |\beta(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x-l)| > t\}|\,dx \\
&= \int_{\mathbb{R}^N} |\{v : |\sum_{l\in\mathbb{Z}^N} \beta(l)|k * F(.,x-l)(v)| > t\}|\,dx, \quad \textit{where } F(v,x) = T_v f(x).
\end{aligned}$$

We know that $\sup_{t>0} t\lambda_f(t)^{\frac{1}{p}} = ||f||_{L^{p,\infty}}$ for $f \in L^{p,\infty}$. Also, since $p > 1$, as stated in §5 in Chapter 1, $||\ ||_{p,\infty}$ is equivalent to the norm $||\ ||^*_{p,\infty}$ ([39]), we have

$$\begin{aligned}
\lambda(E_t) &\leq \int_{\mathbb{R}^N} \frac{1}{t^p} ||\sum_{l\in\mathbb{Z}^N} \beta(l)|k * F(.,x-l)||^p_{L^{p\infty}(\mathbb{T}^N)}\,dx \\
&\leq C_p^p \int_{\mathbb{R}^N} \frac{1}{t^p} (\sum_{l\in\mathbb{Z}^N} \beta(l)||k * F(.,x-l)||^*_{L^{p\infty}(\mathbb{T}^N)})^p\,dx, \quad \text{where } C_p = \frac{p}{p-1} \\
&\leq C_p^p \int_{\mathbb{R}^N} \frac{1}{t^p} (\sum_{l\in\mathbb{Z}^N} \beta(l)||k * F(.,x-l)||^*_{p\infty})^p\,dx \\
&\leq C_p \int_{\mathbb{R}^N} \frac{1}{t^p} (\sum_{l\in\mathbb{Z}^N} \beta(l) N_p^{(w)}(k) ||F(.,x-l)||_{L^p(\mathbb{T}^N)})^p\,dx,
\end{aligned}$$

where $N_p^{(w)}(k)$ is the weak-type norm of the convolution operator $f \longmapsto k*f$ for $f \in L^p(\mathbb{T}^N)$.

Thus,

$$\begin{aligned}
\lambda(E_t) &\leq C_p^p \frac{1}{t^p} \sum_{l\in\mathbb{Z}^N} \beta(l) N_p^{(w)}(k) (\int_{\mathbb{R}^N}\int_{\mathbb{T}^N} |T_v f(x-l)|^p\,dx\,dv)^{\frac{1}{p}})^p \\
&= C_p^p \frac{1}{t^p} (\sum_{l\in\mathbb{Z}^N} \beta(l) N_p^{(w)}(k) (\int_{\mathbb{T}^N}\int_{\mathbb{R}^N} |T_v f(x-l)|^p\,dx\,dv)^{\frac{1}{p}})^p \\
&\leq C_p^p \frac{1}{t^p} (\sum_{l\in\mathbb{Z}^N} \beta(l) N_p^{(w)}(k) C_T ||f||_p)^p \\
&= (\frac{CC_pC_T}{t^p} N_p^{(w)}(k) ||f||_p)^p.
\end{aligned}$$

Hence, $H_k f$ satisfies a weak $(p,p)$ inequality.

### 4.3.3 The Main Theorem

In order to prove the weak-type analogue of Theorem 4.9 we need the following Lemma proved by Asmar, Berkson, and Gillespie in [4].

**Lemma 4.2** [4] *Suppose that* $1 \leq p < \infty$ , $\{\phi_j\} \subseteq M_p^{(w)}(\hat{G})$; $\sup_j\{|\phi_j(\gamma)| : j \in \mathbb{N}, \gamma \in \hat{G}\} < \infty$ *and suppose* $\phi_j$ *converges pointwise a.e. on* $\hat{G}$ *to a function* $\phi$ . *If* $\liminf_j N_p^{(w)}(\phi_j) < \infty$ *then* $\phi \in M_p^{(w)}(\hat{G})$ *and* $N_p^{(w)}(\phi) \leq \liminf_j N_p^{(w)}(\phi_j)$.

**Theorem 4.11** *Suppose* $1 < p < \infty$ *and* $\Lambda \in M_p(\hat{\mathbb{R}^N})$ *is supported in the set* $[\frac{1}{4}, \frac{3}{4}]^N$. *For* $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ *define*

$$W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}^N} \phi(n)\Lambda(\xi - n) \quad \text{on } \mathbb{R}^N.$$

*Then* $W_{\phi,\Lambda} \in M_p^{(w)}(\hat{\mathbb{R}}^N)$ *and* $N_p^{(w)}(W_{\phi,\Lambda}) \leq C N_p^{(w)}(\phi)\|\Lambda\|_{M_p(\mathbb{R}^N)}$.

**Proof:** With the help of Lemma 4.2 we first show that it is enough to prove the theorem for $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ having finite support. Suppose the theorem is true for finitely supported $\phi$. Then for arbitrary $\phi \in M_p^{(w)}(\mathbb{Z}^N)$, define $\phi_j = \hat{k}_j\phi$, where $k_j$ is the j-th Fejér kernel. Then for each $j$ , $\phi_j$'s have finite support and $(T_{\phi_j}f)^\wedge(n) = \phi_j(n)\hat{f}(n) = (T_\phi(k_j * f))^\wedge(n)$. So $\phi_j \in M_p^{(w)}(\mathbb{Z}^N)$ for each $j$ and $N_p^{(w)}(\phi_j) \leq N_p^{(w)}(\phi)$. Define $W_{\phi_j,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}^N} \phi_j(n)\Lambda(\xi - n)$. Now $\liminf_j W_{\phi_j,\Lambda}(\xi) = W_{\phi,\Lambda}(\xi)$. Also, by our assumption

$$\begin{aligned} N_p^{(w)}(W_{\phi_j,\Lambda}) &\leq C N_p^{(w)}(\phi_j)\|\Lambda\|_{M_p(\hat{\mathbb{R}}^N)} \\ &\leq C N_p^{(w)}(\phi)\|\Lambda\|_{M_p(\hat{\mathbb{R}}^N)} \end{aligned}$$

and $|W_{\phi_j,\Lambda}| \leq 2\|\Lambda\|_\infty\|\phi_j\|_\infty \leq 2\|\Lambda\|_\infty\|\phi\|_\infty$. Thus by Lemma 4.2, applied to $W_{\phi_j,\Lambda}$'s ,we conclude that $W_{\phi,\Lambda} \in M_p^{(w)}(\hat{\mathbb{R}}^N)$. Hence it is enough to assume that $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ has finite support.

Now let $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ be finitely supported. Define $k(u) = \sum_{n\in\mathbb{Z}^N} \phi(n)e^{-2\pi i u.n}$ then $k \in L^1(\mathbb{T}^N)$ and $\hat{k}(n) = \phi(n)$. For this particular $k$ and the transference couple $(S, T)$

defined above. We have

$$\begin{aligned}(H_k f)^\wedge(\xi) &= \int_{\mathbb{T}^N} k(u)(T_{u^{-1}}f)^\wedge(\xi)du \\ &= \int_{\mathbb{T}^N} k(u)\sum_{n\in\mathbb{Z}^N}\Lambda(\xi-n)e^{-2\pi iu.n}\hat{f}(\xi)du \\ &= \sum_{n\in\mathbb{Z}^N}\Lambda(\xi-n)\hat{k}(n)\hat{f}(\xi) \\ &= \sum_{n\in\mathbb{Z}^N}\phi(n)\Lambda(\xi-n)\hat{f}(\xi) \\ &= (T_{W_{\phi,\Lambda}}f)^\wedge(\xi).\end{aligned}$$

Thus $T_{W_{\phi,\Lambda}}f = H_k f$. Hence from Theorem 4.10 we have

$$\lambda\{x\in\mathbb{R}^N : |T_{W_{\phi,\Lambda}}f(x)| > t\} \leq (\frac{C}{t}N_p^{(w)}(\phi)||\Lambda||_{M_p(\hat{\mathbb{R}}^N)}||f||_p)^p.$$

### 4.3.4 Lattice Preserving Linear Transformations and Multipliers

We shall now relax the hypothesis that *supp* $\Lambda \subseteq [\frac{1}{4},\frac{3}{4}]^N$ to allow $\Lambda$ to have arbitrary compact support. In fact this can be done by a partition of identity argument as in [11]. Here we give a different method by proving Lemma 4.5 below. Particular cases of this lemma occur in [29] and in [5]. Suppose *supp* $\Lambda \subseteq [-M,M]^N$; define $\Lambda_M(\xi) = \Lambda_1(4M\xi)$, where $\Lambda_1(\xi) = \Lambda(\xi - \frac{1}{2})$. So *supp* $\Lambda_M \subseteq [\frac{1}{4},\frac{3}{4}]^N$. Thus if we define a non-singular transformation $A : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ such that $Ax = 4Mx$ then $\Lambda_M = \Lambda_1 \circ A$. In order to replace the support condition we need to prove $\Lambda_M \circ A^{-1}$ is a summability kernel. In the work of Jodeit and of Asmar, Berkson and Gillespie they assume $A$ in Lemma 4.5 to be multiplication by 2. We have combined some of the results proved by Gröchenig and Madych in [27] in the following lemma which will help us to prove Lemma 4.4. In the proof of Theorem 4.12, we only use the case of a diagonal linear transform, but the more general results proved below are of some interest in their own right.

**Lemma 4.3** [27] *Let* $A : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *be a non-singular linear transformation which preserves the lattice* $\mathbb{Z}^N$ *(i.e.* $A(\mathbb{Z}^N) \subseteq \mathbb{Z}^N$*). Then the following are true.*

*(i) The number of distinct coset representatives of* $\mathbb{Z}^N/A\mathbb{Z}^N$ *is equal to* $q = |\det A|$.

*(ii) If* $Q_0 = [0,1)^N$ *and* $k_1, \ldots\ldots, k_q$ *are the distinct coset representatives of* $\mathbb{Z}^N/A\mathbb{Z}^N$ *then the sets* $A^{-1}(Q_0 + k_i)$ *are mutually disjoint.*

*(iii) Let* $Q = \cup_{i=1}^{q} A^{-1}(Q_0 + k_i)$, *then* $\lambda(Q) = 1$ *and* $\cup_{k\in\mathbb{Z}^N}(Q + k) \simeq \mathbb{R}^N$.

*(iv)* $AQ \simeq \cup_{i=1}^{q}(Q_0 + k_i)$.

The above result is essentially contained in [27]. For convenience we state it here as a lemma and give the details of the proof. We need the following lemma whose proof is easy and can be found in [27].

**Lemma 4.4** *Suppose* $Q$ *is a measurable subset of* $\mathbb{R}^N$ *such that* $\cup_{k\in\mathbb{Z}^N}(Q + k) \simeq \mathbb{R}^N$. *Then the following are equivalent.*

*(1)* $Q \cap (Q + k) = \phi$, *where* $k \in \mathbb{Z}^N \setminus \{0\}$.

*(2)* $\lambda(Q) = 1$.

**Proof of Lemma 4.3:** Let $Q_0 = [0,1)^N$ and $k_1, \ldots, k_m$ be the distinct coset representatives of $\mathbb{Z}^N/A\mathbb{Z}^n$. Then,

$$\begin{aligned}
\mathbb{R}^n &= \cup_{k\in\mathbb{Z}^N}(Q_0 + k) \\
&= \cup_{k\in\mathbb{Z}^N} \cup_{i=1}^{m} (Q_0 + k_i + Ak) \\
&= \cup_{k\in\mathbb{Z}^N}\{Ak + \cup_{i=1}^{m}(Q_0 + k_i).
\end{aligned}$$

Since $A^{-1}\mathbb{R}^N = \mathbb{R}^N$, applying $A^{-1}$ to both sided, we get

$$\mathbb{R}^N = \cup_{k\in\mathbb{Z}^N}\{k + \cup_{i=1}^{m} A^{-1}(Q_0 + k_i)\} = \cup_{k\in\mathbb{Z}^N}(Q + k),$$

where $Q = \cup_{i=1}^{m} A^{-1}(Q_0 + k_i)$. So, $Q$ satisfies the hypothesis and (1) of the Lemma 4.4. Hence $\lambda(Q) = 1$. Since $Q$ is the union of disjoint subsets $A^{-1}(Q_0 + k_1), \ldots, A^{-1}(Q_0 + k_m)$, each having measure $\frac{1}{q}$, it follows that $m = q$. So we have proved (i). Proof of (ii) is trivial. As $m = q$, the proof of (iii) is included in the proof of (i). (iv) follows from (iii).

**Lemma 4.5** *Let $A$ be as in Lemma 4.3. Denote $A^t = B$. For $\phi \in l_\infty(\mathbb{Z}^N)$ define*

$$\psi(n) = \phi(Bn)$$

*and*

$$\eta(n) = \begin{cases} \phi(B^{-1}n) & n \in B\mathbb{Z}^N \\ 0 & \text{otherwise.} \end{cases}$$

*(i) If $\phi \in M_p(\mathbb{Z}^N)$ then $\psi, \eta \in M_p(\mathbb{Z}^N)$ with multiplier norms not exceeding the multiplier norm of $\phi$.*

*(ii) If $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ then $\psi, \eta \in M_p^{(w)}(\mathbb{Z}^N)$ with weak multiplier norms not exceeding the weak multiplier norm of $\phi$.*

**Proof:** (i) For $f \in L^p(Q_0)$, we let $f$ again denote the periodic extension to $\mathbb{R}^N$. Define $Sf(x) = f(Ax)$ , then $Sf$ is also periodic and

$$\begin{aligned} \int_{Q_0} |Sf(x)|^p dx &= \int_{Q_0} |Sf(x)|^p \sum_j \chi_Q(x-j) dx \\ &= \sum_j \int_{Q_0+j} |Sf(x)|^p \chi_Q(x) dx \\ &= \int_Q |Sf(x)|^p dx \\ &= \frac{1}{|\det A|} \int_{AQ} |f(x)|^p dx \\ &= \frac{1}{q} \sum_{i=1}^{q} \int_{Q_0+k_i} |f(x)|^p dx \qquad ((iv) \text{ of Lemma 4.3}) \\ &= \int_{Q_0} |f(x)|^p dx. \end{aligned}$$

Thus $S$ is an isometry, i.e., $\| Sf \|_{L^p(Q_0)} = \| f \|_{L^p(Q_0)}$.

Further,

$$\begin{aligned}
(Sf)^\wedge(n) &= \int_{Q_0} f(Ax)e^{-2\pi i x.n}dx \\
&= \frac{1}{q}\int_{AQ_0} f(x)e^{-2\pi i A^{-1}x.n}dx \\
&= \frac{1}{q}\sum_{i=1}^{q}\int_{Q_0+k_i} f(x)e^{-2\pi i x.B^{-1}n}dx \\
&= \frac{1}{q}\sum_{i=1}^{q}\int_{Q_0} f(x)e^{-2\pi i(x+k_i).B^{-1}n}dx \\
&= \frac{1}{q}\sum_{i=1}^{q}\int_{Q_0} f(x)e^{-2\pi i(x+k_i).(m+B^{-1}l_j)}dx \\
&= \frac{1}{q}\int_{Q_0} f(x)e^{-2\pi i x(m+B^{-1}l_j)}\sum_{i=1}^{q} e^{-2\pi i k_i.B^{-1}l_j}dx,
\end{aligned}$$

(where $l_1, \ldots, l_q$ are distinct coset representatives of $\mathbb{Z}^N/B\mathbb{Z}^N$). So, from the orthogonality relations of the characters (Lemma 1, [33]) we have

$$(Sf)^\wedge(n) = \begin{cases} \hat{f}(B^{-1}n) & if\ n \in B\mathbb{Z}^N \\ 0 & otherwise. \end{cases}$$

For $f \in L^p(Q_0)$ we define an operator on $L^p(Q_0)$ such that $Wf(x) = \frac{1}{q}\sum_{i=1}^{q} f(A^{-1}(x+\ k_i))$, where $k_1, \ldots, k_q$ are distinct cosets of $\mathbb{Z}^N \setminus A\mathbb{Z}^N$. Then for a trigonometric polynomial $f$,

$$\begin{aligned}
(Wf)^\wedge(n) &= \frac{1}{q}\sum_{i=1}^{q}\int_{Q_0} f(A^{-1}(x+k_i))e^{-2\pi i n.x}dx \\
&= \frac{1}{q}\sum_{i=1}^{q}\int_{Q_0+k_i} f(A^{-1}x)e^{-2\pi i n.x}dx \\
&= \sum_{i=1}^{q}\int_{A^{-1}(Q_0+k_i)} f(x)e^{-2\pi i A^t n.x}dx \\
&= \hat{f}(Bn),
\end{aligned}$$

and so

$$
\begin{aligned}
(\int_{Q_0} |Wf(x)|^p dx)^{\frac{1}{p}} &= (\int_{Q_0} |\frac{1}{q}\sum_{i=1}^{q} f(A^{-1}(x+k_i))|^p dx)^{\frac{1}{p}} \\
&\le \frac{1}{q}\sum_{i=1}^{q}(\int_{Q_0} |f(A^{-1}(x+k_i))|^p dx)^{\frac{1}{p}} \\
&= \frac{1}{q}\sum_{i=1}^{q}(\int_{Q_0+k_i} |f(A^{-1}x)|^p dx)^{\frac{1}{p}} \\
&= \frac{q^{\frac{1}{p}}}{q}\sum_{i=1}^{q}(\int_{A^{-1}(Q_0+k_i)} |f(x)|^p dx)^{\frac{1}{p}} \\
&= \frac{q^{\frac{1}{p}}}{q}(\int_{Q} |f(x)|^p dx)^{\frac{1}{p}} \\
&= q^{\frac{1-p}{p}} \parallel f \parallel_{L^p(Q_0)} .
\end{aligned}
$$

Therefore $\parallel Wf \parallel_{L^p(Q_0)} \le q^{\frac{1-p}{p}} \parallel f \parallel_{L^p(Q_0)}$. It is easy to see that

$$ST_\phi W = T_\eta \tag{4.13}$$

and

$$WT_\phi S = T_\psi \tag{4.14}$$

It follows that, if $\phi \in M_p(\mathbb{Z}^N)$ then $\|T_\psi f\| \le C_p\|\phi\|_{M_p(\mathbb{Z}^N)}\|f\|_{L^p(Q_0)}$ Also $\|T_\eta f\|_{L^p(Q_0)} \le C_p\|\phi\|_{M_p(\mathbb{Z}^N)}\|f\|_{L^p(Q_0)}$. Hence $\psi, \eta \in M_p(\mathbb{Z}^N)$.

(ii) For $\phi \in M_p^{(w)}(\mathbb{Z}^N)$, we need to calculate the distribution function of $Sf$ and $Wf$. Denote $E_t = \{x \in Q_0 : |Sf(x)| > t > 0\}$. Then

$$
\begin{aligned}
|E_t| &= \int_{Q_0} \chi_{E_t}(x)dx \\
&= \int_{Q_0} \chi_{\mathbb{R}_+}(|f(Ax)| - t)dx \\
&= \int_{Q} \chi_{\mathbb{R}_+}(|f(Ax)| - t)dx \\
&= \frac{1}{q}\int_{AQ} \chi_{\mathbb{R}_+}(|f(x)| - t)dx \\
&= \frac{1}{q}\sum_{i=1}^{q}\int_{Q_0+k_i} \chi_{\mathbb{R}_+}(|f(x)| - t)dx
\end{aligned}
$$

$$\begin{aligned} &= \int_{Q_0} \chi_{\mathbb{R}_+}(|f(x)| - t)dx \\ &= |\{x : |f(x) > t\}|. \end{aligned}$$

Therefore

$$|\{x \in Q_0 : |Sf(x) > t\}| = |\{x \in Q_0 : |f(x) > t\}| \tag{4.15}$$

Also

$$\begin{aligned} |\{x \in Q_0 : |Wf(x)| > t\}| &= |\{x \in Q_0 : |\sum_{i=1}^{q} f(A^{-1}(x + k_i))| > tq\}| \\ &\leq |\{x \in Q_0 : \sum_{i=1}^{q} |f(A^{-1}(x + k_i))| > tq\}| \\ &= \sum_{i=1}^{q} \int_{Q_0} \chi_{\mathbb{R}_+}(|f(A^{-1}(x + k_i))| - t)dx \\ &= \sum_{i=1}^{q} \int_{Q_0 + k_i} \chi_{\mathbb{R}_+}(|f(A^{-1}x)| - t)dx \\ &= q\sum_{i=1}^{q} \int_{A^{-1}(Q_0 + k_i)} \chi_{\mathbb{R}_+}(|f(x)| - t)dx. \end{aligned}$$

Thus

$$|\{x \in Q_0 : |Wf(x)| > t\} \leq q|\{x \in Q_0 : |f(x)| > t\}|. \tag{4.16}$$

From the relations (4.13),(4.14) along with (4.15),(4.16), we conclude that $\psi, \eta \in M_p^{(w)}(\mathbb{Z}^N)$ whenever $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ . Also $N_p^{(w)}(\psi) \leq CN_p^{(w)}(\phi)$ and $N_p^{(w)}(\eta) \leq CN_p^{(w)}(\phi)$.

As an application of this Lemma we get the following result regarding weak summability kernels.

**Lemma 4.6** *Let $A$ be as in Lemma 4.3. Suppose $\Lambda$ is a weak (strong) summabiliy kernel then $\Lambda \circ B$ and $\Lambda \circ B^{-1}$ are also weak (strong) summability kernels.*

**Proof:** Define $W_{\phi,\Lambda\circ B}$ on $\mathbb{R}^N$ for $\phi \in M_p^{(w)}(\mathbb{Z}^N)$.

$$\begin{aligned} W_{\phi,\Lambda\circ B}(x) &= \sum_{n\in\mathbb{Z}^N} \phi(n)\Lambda \circ B(x - n) \\ &= \sum_{n\in\mathbb{Z}^N} \phi(n)\Lambda(Bx - Bn) \end{aligned}$$

$$\begin{aligned} &= \sum_{n \in B\mathbb{Z}^N} \phi(B^{-1}n)\Lambda(Bx - n) \\ &= \sum_{n \in \mathbb{Z}^N} \eta(n)\Lambda(Bx - n) \\ &= W_{\eta,\Lambda}(Bx). \end{aligned}$$

As $\eta \in M_p^{(w)}(\mathbb{Z}^N)$ (by Lemma 4.5) and since $\Lambda$ is a summability kernel we have $W_{\eta,\Lambda} \in M_p^{(w)}(\hat{\mathbb{R}}^N)$. Hence $W_{\phi,\Lambda\circ B} \in M_p^{(w)}(\hat{\mathbb{R}}^N)$. Similarly

$$\begin{aligned} W_{\phi,\Lambda\circ B^{-1}} &= \sum_{n \in \mathbb{Z}^N} \phi(n)\Lambda(B^{-1}x - B^{-1}n) \\ &= \sum_{n \in B\mathbb{Z}^N} \phi(n)\Lambda(B^{-1}x - B^{-1}n) + \cdots + \sum_{n \in B\mathbb{Z}^N + p_{q-1}} \phi(n)\Lambda(B^{-1}x - B^{-1}n) \end{aligned}$$

where $p_1 \ldots . p_{q-1}$ are distinct coset representatives of $B\mathbb{Z}^N \setminus \mathbb{Z}^N$

$$\begin{aligned} W_{\phi,\Lambda\circ B^{-1}} &= \sum_{n \in \mathbb{Z}^N} \phi(Bn)\Lambda(B^{-1}x - n) + \ldots + \sum_{n \in \mathbb{Z}^N} \phi(Bn + p_{q-1})\Lambda(B^{-1}x + B^{-1}p_{q-1} - n) \\ &= W_{\psi,\Lambda}(B^{-1}x) + \ldots + W_{\psi_{p_{q-1}},\Lambda}(B^{-1}x + B^{-1}p_{q-1}) \end{aligned}$$

where $\psi_{p_i}(l) = (\tau_{p_i}\phi)(Bl)$, $i = 1, 2, .., q-1$. As $\psi \in M_p^{(w)}(\mathbb{Z}^N)$ and $\Lambda$ is a summability kernel we conclude that $W_{\phi,\Lambda\circ B^{-1}} \in M_p^{(w)}(\hat{\mathbb{R}}^N)$.

Hence from Lemma 4.6 and the discusssion preceeding Lemma 4.3 we conclude the following theorem.

**Theorem 4.12** *Suppose $\Lambda \in M_p(\hat{\mathbb{R}^N})$ and supp $\Lambda \subseteq [-M, M]$; for $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ define $W_{\phi,\Lambda}(\xi) = \sum_{n \in \mathbb{Z}^N} \phi(n)\Lambda(\xi - n)$ on $\mathbb{R}^N$, then $W_{\phi,\Lambda} \in M_p^{(w)}(\hat{\mathbb{R}}^N)$ and $N_p^{(w)}(W_{\phi,\Lambda}) \leq C_\Lambda N_p^{(w)}(\phi)||\Lambda||_{M_p(\mathbb{R}^N)}$, where $C_\Lambda$ is constant depending on $\Lambda$.*

### 4.3.5 An Application

An application of Theorem 4.12 is a weak-type version of a result proved by de Leeuw [39].

**Theorem 4.13** *For $1 < p < \infty$ , and $\epsilon > 0$; let $\{\phi_\epsilon\} \subseteq M_p^{(w)}(\mathbb{Z})$ satisfy*

*(i)* $\lim_{\epsilon \to 0} \phi_\epsilon([\frac{x}{\epsilon}]) = \phi(x)$ *a.e.*

*(ii)* $\sup_\epsilon N_p^{(w)}(\phi_\epsilon) = K < \infty$.

*Then $\phi \in M_p^{(w)}(\hat{\mathbb{R}})$ and $N_p^{(w)}(\phi) \leq \sup_\epsilon N_p^{(w)}(\phi_\epsilon)$.*

**Proof:** for each $\epsilon > 0$, define $W_{\phi_\epsilon}$ on $\mathbb{R}$ by

$$W_{\phi_\epsilon}(x) = \sum_{n \in \mathbb{Z}} \phi_\epsilon(n) \chi_{[0,1)}(x - n). \tag{4.17}$$

As $\chi_{[0,1)} \in M_p(\hat{\mathbb{R}})$ for $1 < p < \infty$, from Theorem 4.12 we have $W_{\phi_\epsilon} \in M_p^{(w)}(\hat{\mathbb{R}})$ and $N_p^{(w)}(W_{\phi_\epsilon} \leq C N_p^{(w)}(\phi_\epsilon) \leq CK$ . We define another function $\psi_\epsilon$, for each $\epsilon > 0$, by $\psi_\epsilon(x) = W_{\phi_\epsilon}(\frac{x}{\epsilon})$. Then $\psi_\epsilon \in M_p^{(w)}(\hat{\mathbb{R}})$ and

$$N_p^{(w)}(\psi_\epsilon) \leq N_p^{(w)}(W_{\phi_\epsilon}) \leq Ck. \tag{4.18}$$

From (4.17) we have

$$\begin{aligned} \psi_\epsilon(x) \;=\; W_{\phi_\epsilon}(\frac{x}{\epsilon}) &= \sum_{n \in \mathbb{Z}} \phi_\epsilon(n) \chi_{[0,1)}(\frac{x}{\epsilon} - n) \\ &= \phi_\epsilon([\frac{x}{\epsilon}]). \end{aligned}$$

So from our hypothesis

$$\lim_{\epsilon \to 0} \psi_\epsilon(x) = \phi(x) \quad a.e. \tag{4.19}$$

Also we have $|\psi_\epsilon(x)| < \infty$ $\quad$ (as $\sup_{\epsilon,n} |\phi_\epsilon(n)| < \infty$).

Hence from (4.17), (4.18) and (4.19) along with Lemma 4.2 we have $\phi \in M_p^{(w)}(\hat{\mathbb{R}})$ and $N_p^{(w)}(\phi) \leq \lim_\epsilon N_p^{(w)}(\phi_\epsilon) \leq CK$.

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